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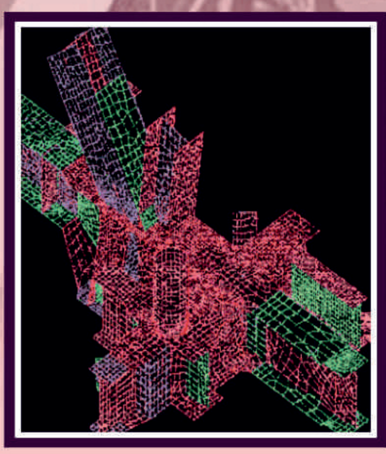
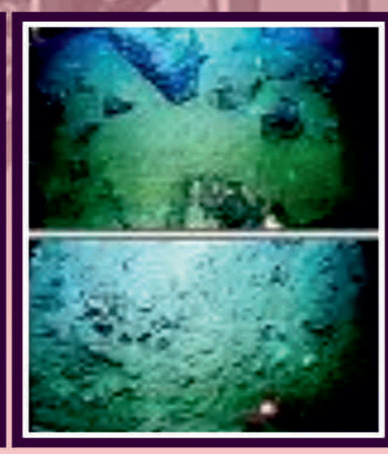
THEORY OF PLATES AND SHELLS





S.S. Bhavikatti

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AND
SHELLS

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THEORY OF PLATES AND SHELLS

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Preface

Plate is a flat surface having considerably large dimensions compared to its thickness with supports along few edges and is subjected to transverse load. For civil engineer common example of plate is a slab. At undergraduate level students are taught design of slab by approximate methods or by using moment co-efficients given in the code, without going through how they are obtained. At the post-graduate level theory of plates is taught to structural engineering students to understand actual load transfer in plate by elastic analysis. It involves forming and solving fourth order differential equations.

Shells are curved plates. The analysis of shells involves additional complexity. A design engineer should understand mechanism of load transfer and internal forces developed in the shells. Shells are to cover large area free of columns and architects prefer them for their aesthetic appeal. A structural engineer has to learn theory of shells to design economical shell structures with more confidence.

In this book theory of plates and shells is explained. The analysis is restricted to classical method only. Finite element method is the numerical method suitable for the analysis. Author has covered the shell analysis by finite element analysis in his seperate book. A number of commercial packages are available for the analysis by finite element method. But it is necessary that design engineer should have basic knowledge of load transfer and internal forces that develop, which is possible by going through classical theory. For validating the results obtained by finite element packages, classical theory for commonly found standard cases is essential.

The book is essentially based on the lecture notes of the author taught to students of M.Tech. (Industrial Structure) at NITK Surathkal and M.Tech. (Structural Engineering) at BVB College of Engineering for over 40 years. The author hopes that the book will be quite useful as a textbook for M.Tech. students to gain confidence in taking design of plates and shells.

S.S. Bhavikatti

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Acknowledgement

The author sincerely thanks Dr. A.S. Arya, former Professor of Civil Engineering and Director of School of Earthquake Engineering, IIT Roorkee for introducing him to the subject theory of plates and shells and guiding him for M.Tech. thesis on the analysis of continuous cylindrical shell. He is thankful to NITK Surathkal and BVB College of Engineering and Technology for giving him opportunity to teach this subject to M.Tech. students for several batchestime. Thanks are also due to a number of M.Tech. students who took their dissertation work on the analysis and design of shells and to Smt. T.K. Lakshmy for taking up Ph D work on optimum design of shells and folded plates.

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Chapter 1

Introduction to Plates

Plate is a flat surface having considerably large dimensions as compared to its thickness. Common examples of plates in civil engineering are

1. Slab in a building.
2. Base slab and wall of water tanks.
3. Stem of retaining wall.

A plate may have different shapes *e.g.* rectangular, triangular, elliptic, circular etc. as shown in Fig. 1.1.

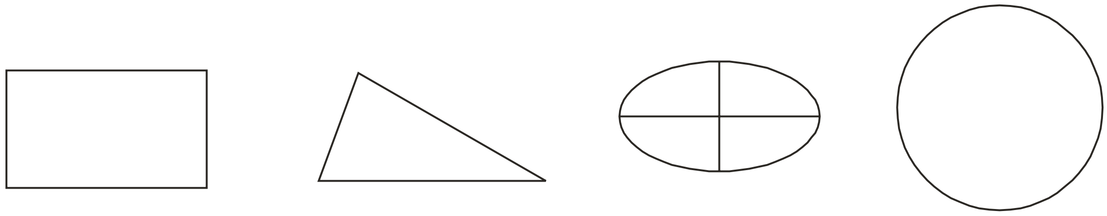


Fig. 1.1 Shapes of plates

A plate may have edge conditions like free, simply supported, fixed or elastically supported as shown in Fig. 1.2.

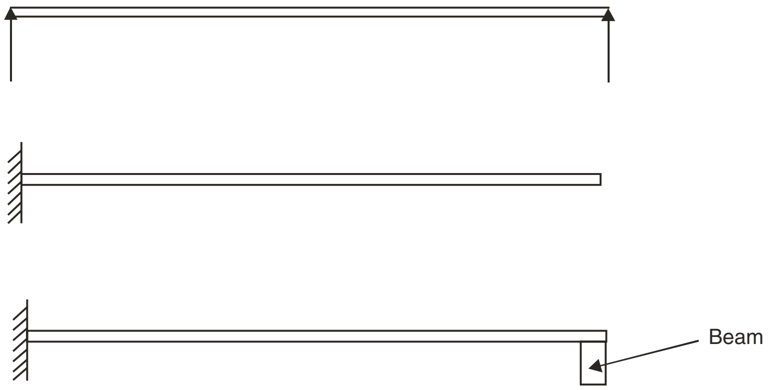


Fig. 1.2 Edge conditions

In this chapter the coordinate system selected is clearly explained first and then various forces to be considered on an element of plate are explained and sign conventions are made clear. At the end a brief introduction is given to different theories available for the analysis of plates.

1.1 COORDINATE SYSTEMS

In the analysis of plates, cartesian coordinate system with right hand rule is used. According to this when thumb, index finger and middle finger are stretched to show three mutually perpendicular directions, thumb indicates *x*-coordinate direction, index finger shows *y*-coordinate direction and middle finger indicates *z*-coordinate direction. Figure 1.3 shows different orientation of *x, y, z* directions. The equations derived for plate analysis with cartesian coordinate system with right hand rule hold good for all these orientations. The commonly used orientation is that shown in Fig. 1.3(*a*), since slabs are usually subjected to downward loads and the analyst is interested in downward deflections.

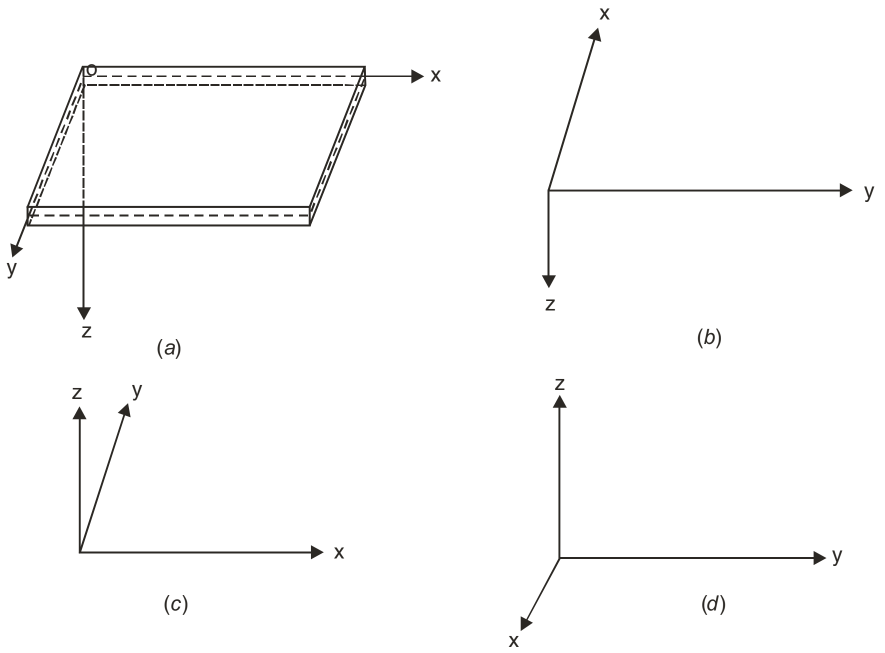


Fig. 1.3 Different orientation of coordinates with right hand rule

For the analysis of circular plates, polar coordinate system shown in Fig. 1.4 may be used advantageously.

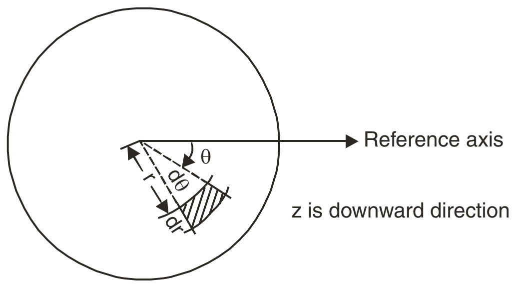
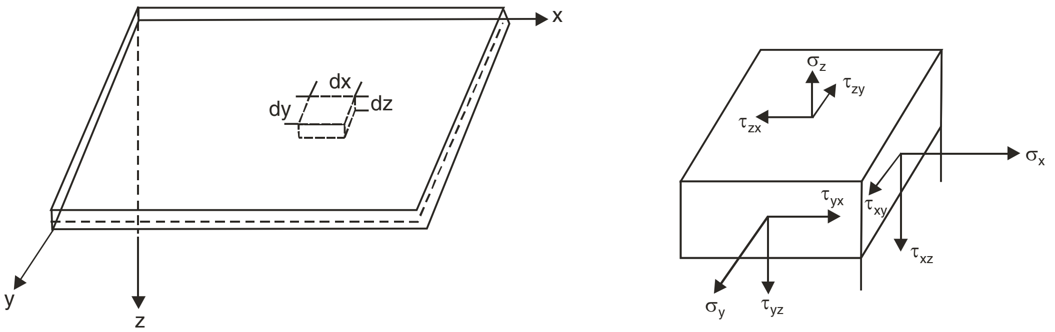


Fig. 1.4 Polar coordinates for circular plate

1.2 STRESSES ON AN ELEMENT

In cartesian system an element of size $dx \times dy \times dz$ is selected at a point (x, y), distance z below the middle surface [Refer Fig. 1.5(*a*)]. The stresses acting on the element are shown in Fig. 1.5(*b*), in their positive senses. Note that the sign convention used is that a stress on positive face in positive direction or on negative face in negative direction is positive stress. It means the direct tensile stress is positive. For shear stresses, the positive senses are as shown in Fig. 1.5(*b*).



(*a*) An element at point (*x, y, z*)

(*b*) Stresses on the element

Fig. 1.5 Stresses on an element

1.3 TYPES OF THEORIES OF PLATES

The theories that are available for the analysis of plates are

1. Thin plates with small deflections.
2. Thin plates with large deflections and
3. Thick plates.

1.3.1 Theory of Thin Plates with Small Deflections

This theory is satisfactory for plates with thickness less than $\frac{1}{20}$th of its lateral dimension and having deflection less than $\frac{1}{5}$th of its thickness. In this theory the following three assumptions are made:

1. Points on the plate lying initially on a normal to the middle surface of the plate remain on the normal to the middle surface of the plate even after bending.
2. The normal stresses in the direction transversal to the plate can be neglected *i.e.* Take σ_z, τ_{xz}, $\tau_{yz} = 0$.
3. There is no deformation in the middle surface of the plate. This plane remains neutral during bending.

Assumption 1 means shear deformations are neglected. This assumption is generally satisfactory, but in some cases *e.g.* in case of holes in the plate, the effect of shear becomes considerable and hence corrections to the theory of thin plates are to be applied.

Assumption 2 is valid for thin plates, since the stresses are zero in z-direction at top and bottom of plates, as they are free edges. There may be small variation inside the plate at any depth z, but it is negligible.

Assumption 3 holds good if the plate is thin. However in actual structure when the plate bends, small forces may develop in the middle surface. This inplane stress in the middle of plate reduces the bending moment at any other point. Hence neglecting this force is an assumption on safer side.

1.3.2 Theory of Thin Plates with Large Deflections

If the deflections are not small in comparison with its thickness, strains and stresses are introduced in the middle surface of the plate. These stresses are to be considered in deriving equilibrium equations. Inclusion of these stresses results into non-linear equations. This is called geometric nonlinearity. When this non-linearity is considered, the solution becomes more complicated.

1.3.3 Theory of Thick Plates

The first two theories discussed above become unrealistic in the case of plates of larger thicknesses, especially in the case of highly concentrated loads. In such cases thick plate theory should be used. This theory considers analysis as a three dimensional problem of elasticity. The analysis becomes lengthy and more complicated. Till today the problems are solved only for a few particular cases.

QUESTIONS

1. Draw an element of plate in Cartesian system and show the stresses acting on it in their positive senses. Make the sign convention clear.
2. Briefly write on the following theories of plates to bring out differences among them.
   (*a*) Thin plates with small deflections.
   (*b*) Thin plates with large deflections.
   (*c*) Thick plates.

Chapter

2

Pure Bending of Plates

As the title suggests, in this theory stress resultants produced due to bending moments only are considered. In other words, deformation of the membrane due to external loads is ignored. Naturally, in this type of bending, middle surface remains neutral surface. In this chapter, some of the properties of bent surface are discussed and expressions are derived for stresses and moments in terms of single unknown deflection '*w*'.

2.1 SLOPE IN SLIGHTLY BENT PLATE

Figure 2.1(*a*) shows the plan view of an element and Fig. 2.1(*b*) shows sectional view of slightly bent plate.

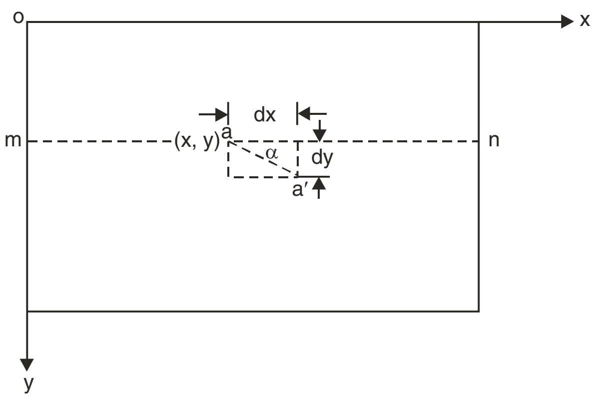


Fig. 2.1 (*a*) Plan view of element

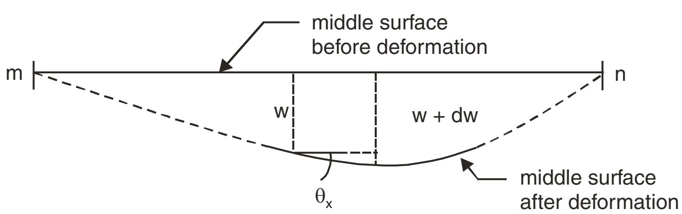


Fig. 2.1 (*b*) Sectional view of element

Consider an element of size $dx \times dy$ at point (x, y) in the middle surface of the plate. Figure 2.1(*b*) shows the middle surface of the plate cut by plane *mn* parallel to *xz* plane. Then,

$$\text{Slope along } x\text{-axis} = \theta_x = \frac{\partial w}{\partial x} \qquad \text{...eqn. 2.1}$$

Similarly if a plane parallel to *yz* plane is considered,

$$\text{Slope along } y\text{-axis} = \theta_y = \frac{\partial w}{\partial y} \qquad \text{...eqn. 2.2}$$

Let aa' make an angle α with x-axis (Refer Fig. 2.1(*a*)). The difference between the deflections at a and a' is due to slopes in x and y directions. Let it be 'dw'.

Then,

$$dw = \theta_x\, dx + \theta_y\, dy$$

$$= \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy \qquad \text{...eqn. 2.3}$$

Slope along aa' which is in 'n' direction is given by

$$\frac{dw}{\partial n} = \frac{\partial w}{\partial x}\frac{dx}{\partial n} + \frac{\partial w}{\partial y}\frac{dy}{dn}$$

$$= \frac{\partial w}{\partial x}\cos\alpha + \frac{\partial w}{\partial y}\sin\alpha. \qquad \text{...eqn. 2.4}$$

Let the maximum slope be at an angle α to x-axis. Hence

$$\frac{\partial}{\partial \alpha}\left(\frac{dw}{dn}\right)\bigg|_{\alpha=\alpha_1} = 0$$

$$\frac{\partial w}{\partial x}(-\sin\alpha_1) + \frac{\partial w}{\partial y}\cos\alpha_1 = 0$$

or

$$\tan\alpha_1 = \frac{\partial w/\partial y}{\partial w/\partial x} \qquad \text{...eqn. 2.5}$$

Putting eqn. 2.4 to zero, we get the direction of zero slope. Let it be α_2. Then

$$0 = \frac{\partial w}{\partial x}\cos\alpha_2 + \frac{\partial w}{\partial y}\sin\alpha_2$$

$\therefore$

$$\tan\alpha_2 = -\frac{\partial w/\partial x}{\partial w/\partial y} \qquad \text{...eqn. 2.6}$$

From eqns. 2.5 and 2.6, we get,

$$\tan\alpha_1 \cdot \tan\alpha_2 = -1.$$

It means **the direction of zero slope (α_2) and the direction of maximum slope (α_1) are at right angles to each other.**

Expression for Maximum Slope:

The value of maximum slope

$$= \left.\frac{\partial w}{\partial n}\right|_{\alpha=\alpha_1}$$

$$= \frac{\partial w}{\partial x}\cos\alpha_1 + \frac{\partial w}{\partial y}\sin\alpha_1$$

$$= \left(\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}\cdot\tan\alpha_1\right)\cos\alpha_1$$

$$= \left(\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial w/\partial y}{\partial w/\partial x}\right)\frac{1}{\sec\alpha_1}$$

$$= \frac{\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2}{\partial w/\partial x}\frac{1}{\sqrt{1+\tan^2\alpha_1}}$$

$$= \frac{\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2}{\partial w/\partial x}\frac{1}{\sqrt{1+\left(\frac{\partial w/\partial y}{\partial w/\partial x}\right)^2}}$$

$$= \frac{\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2}{\partial w/\partial x}\frac{\frac{\partial w}{\partial x}}{\sqrt{\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2}}$$

$$= \sqrt{\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2} \qquad \text{...eqn. 2.7}$$

2.2 CURVATURE OF SLIGHTLY BENT PLATE

The curvature of a bent surface is numerically equal to the rate of change of slope. If the curvature is considered positive when it is a sagging surface (Refer Fig. 2.2), the curvature in x-direction

$$= \frac{1}{r_x} = -\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right) = -\frac{\partial^2 w}{\partial x^2} \qquad \text{...eqn. 2.8}$$

where $\frac{1}{r_x}$ is radius of curvature.

Similarly, curvature in y-direction

$$= \frac{1}{r_y} = -\frac{\partial^2 w}{\partial y^2}. \qquad \text{...eqn. 2.9}$$

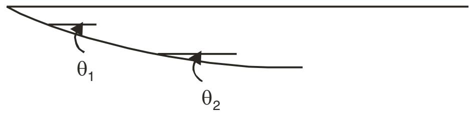


Fig. 2.2 Positive sense of curvature

Curvature in any direction

Consider a direction n which makes angle α with x-axis. Then from the definition of curvature,

$$\frac{1}{r_n} = -\frac{\partial^2 w}{\partial n^2}$$

From eqn. 2.3,

$$\frac{\partial w}{\partial n} = \frac{\partial w}{\partial x}\cos\alpha + \frac{\partial w}{\partial y}\sin\alpha$$

i.e.

$$\frac{\partial}{\partial n} = \frac{\partial}{\partial x}\cos\alpha + \frac{\partial}{\partial y}\sin\alpha$$

$\therefore$

$$\frac{1}{r_n} = -\frac{\partial^2 w}{\partial n^2} = -\frac{\partial}{\partial n}\left(\frac{\partial w}{\partial n}\right)$$

$$= -\left(\frac{\partial}{\partial x}\cos\alpha + \frac{\partial}{\partial y}\sin\alpha\right)\left(\frac{\partial w}{\partial x}\cos\alpha + \frac{\partial w}{\partial y}\sin\alpha\right)$$

$$= -\left[\frac{\partial^2 w}{\partial x^2}\cos^2\alpha + \frac{\partial^2 w}{\partial y^2}\sin^2\alpha + \frac{\partial^2 w}{\partial x \partial y}\cos\alpha\,\sin\alpha + \frac{\partial^2 w}{\partial x\,\partial y}\sin\alpha\cdot\cos\alpha\right]$$

Noting that,

$$\frac{1}{r_x} = -\frac{\partial^2 w}{\partial x^2}$$

and

$$\frac{1}{r_y} = -\frac{\partial^2 w}{\partial y^2}$$

and taking $\frac{1}{r_{xy}} = \frac{\partial^2 w}{\partial x \partial y}$, we get

$$\frac{1}{r_n} = \frac{1}{r_x}\cos^2\alpha + \frac{1}{r_y}\sin^2\alpha - \frac{2\partial^2 w}{\partial x \partial y}\sin\alpha\cos\alpha$$

$$= \frac{1}{r_x}\frac{1+\cos 2\alpha}{2} + \frac{1}{r_y}\frac{1-\cos 2\alpha}{2} - \frac{1}{r_{xy}}\sin 2\alpha$$

$$= \frac{1}{2}\left(\frac{1}{r_x} + \frac{1}{r_y}\right) + \left(\frac{1}{r_x} - \frac{1}{r_y}\right)\frac{\cos 2\alpha}{2} - \frac{1}{r_{xy}}\sin 2\alpha \qquad \text{...eqn. 2.10}$$

If 't' is the direction at right angles to 'n' direction, the direction 't' is at $\alpha + 90°$ to n-direction. Hence $\frac{1}{r_t}$ can be obtained by changing α to $\alpha + 90$ in eqn. 2.9. Thus

$$\frac{1}{r_t} = \frac{1}{2}\left(\frac{1}{r_x} + \frac{1}{r_y}\right) + \frac{1}{2}\left(\frac{1}{r_x} - \frac{1}{r_y}\right)\cos(2\alpha + 90) - \frac{1}{r_{xy}}\sin 2(\alpha + 90)$$

$$= \frac{1}{2}\left(\frac{1}{r_x} + \frac{1}{r_y}\right) - \frac{1}{2}\left(\frac{1}{r_x} - \frac{1}{r_y}\right)\cos 2\alpha + \frac{1}{r_{xy}}\sin 2\alpha \quad \text{...eqn. 2.11}$$

Adding eqns. 2.10 and 2.11, we get

$$\frac{1}{r_n} + \frac{1}{r_t} = \frac{1}{r_x} + \frac{1}{r_y} \quad \text{...eqn. 2.12}$$

Hence we can conclude, **the sum of curvatures in any two mutually perpendicular directions in a slightly bent plate is constant.**

Twist of the surface w.r.t. *n* and *t* directions:

It is given by,

$$\frac{1}{r_{nt}} = \frac{\partial^2 w}{\partial n \partial t} = \frac{\partial}{\partial n}\left(\frac{\partial w}{\partial t}\right)$$

Now from eqn. 2.4,

$$\frac{\partial w}{\partial n} = \frac{\partial w}{\partial x}\cos\alpha + \frac{\partial w}{\partial y}\sin\alpha$$

$\therefore$
$$\frac{\partial}{\partial n} = \frac{\partial}{\partial x}\cos\alpha + \frac{\partial}{\partial y}\sin\alpha$$

Since t is the direction at right angles to n, we get $\frac{\partial}{\partial t}$ from the above expression by changing α to $\alpha + 90$.

i.e.
$$\frac{\partial}{\partial t} = \frac{\partial}{\partial x}\cos(\alpha + 90) + \frac{\partial}{\partial y}\sin(\alpha + 90)$$

$$= -\frac{\partial}{\partial x}\sin\alpha + \frac{\partial}{\partial y}\cos\alpha$$

$\therefore$
$$\frac{1}{r_{nt}} = \frac{\partial}{\partial n}\left(\frac{\partial w}{\partial t}\right)$$

$$= \left(\frac{\partial}{\partial x}\cos\alpha + \frac{\partial}{\partial y}\sin\alpha\right)\left(-\frac{\partial w}{\partial x}\sin\alpha + \frac{\partial w}{\partial y}\cos\alpha\right)$$

$$= -\frac{\partial^2 w}{\partial x^2}\sin\alpha\cos\alpha + \frac{\partial^2 w}{\partial y^2}\sin\alpha\cos\alpha + \frac{\partial^2 w}{\partial x\partial y}\cos^2\alpha - \frac{\partial^2 w}{\partial x\partial y}\sin^2\alpha$$

$$=\left(\frac{1}{r_x}-\frac{1}{r_y}\right)\frac{\sin 2\alpha}{2}+\frac{1}{r_{xy}}\left(\cos^2\alpha-\sin^2\alpha\right)$$

$$=\left(\frac{1}{r_x}-\frac{1}{r_y}\right)\frac{\sin 2\alpha}{2}+\frac{1}{r_{xy}}\cos 2\alpha \qquad \text{...eqn. 2.13}$$

2.3 PRINCIPAL CURVATURE

The two mutually perpendicular directions '*n*' and '*t*' with respect to which twist of the surface $\frac{1}{r_{nt}}=0$, is called the direction of principal curvatures. Hence from eqn. 2.13, we get the direction of principal curvature 'α' as

$$\tan 2\alpha=-\frac{2\times\frac{1}{r_{xy}}}{\left(\frac{1}{r_x}-\frac{1}{r_y}\right)} \qquad \text{...eqn. 2.14}$$

It can be shown that in the direction of principal curvatures, the curvature is maximum/minimum. For this proof, differentiate eqn. 2.10 with respect to α. It gives,

$$\frac{1}{2}\left(\frac{1}{r_x}-\frac{1}{r_y}\right)(-\sin 2\alpha)\times 2-\frac{1}{r_{xy}}2\cos 2\alpha=0$$

i.e.

$$\tan 2\alpha=-\frac{2\times\frac{1}{r_{xy}}}{\left(\frac{1}{r_x}-\frac{1}{r_y}\right)},$$

which is same as eqn. 2.14.

Thus we find the **planes of principal curvatures are the planes of extreme curvatures also.**

Magnitude of Principal Curvatures

For such planes,

$$\tan 2\alpha=-\frac{2\frac{1}{r_{xy}}}{\frac{1}{r_x}-\frac{1}{r_y}}$$

Referring to Fig. 2.3

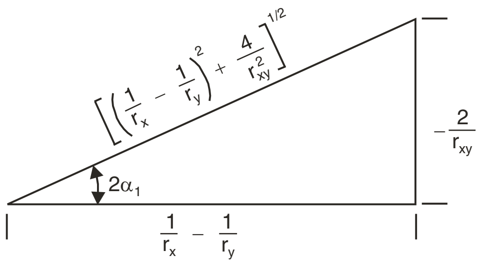


Fig. 2.3

$$\sin 2\alpha = \frac{-2\frac{1}{r_{xy}}}{\left[\left(\frac{1}{r_x}-\frac{1}{r_y}\right)^2+\frac{4}{r_{xy}^2}\right]^{1/2}} \text{ and } \cos 2\alpha = \frac{\frac{1}{r_x}-\frac{1}{r_y}}{\left[\left(\frac{1}{r_x}-\frac{1}{r_y}\right)^2+\frac{4}{r_{xy}^2}\right]^{1/2}}$$

Substituting these values in eqn. 2.10 and noting this as $\frac{1}{r_1}$, we get

$$\frac{1}{r_1} = \frac{1}{2}\left(\frac{1}{r_x}+\frac{1}{r_y}\right)+\left(\frac{1}{r_x}-\frac{1}{r_y}\right)\frac{1}{2}\frac{\frac{1}{r_x}-\frac{1}{r_y}}{\left[\left(\frac{1}{r_x}-\frac{1}{r_y}\right)^2+\frac{4}{r_{xy}^2}\right]^{1/2}} - \frac{1}{r_{xy}}\frac{\left(-2\frac{1}{r_{xy}}\right)}{\left[\left(\frac{1}{r_x}-\frac{1}{r_y}\right)^2+\frac{4}{r_{xy}^2}\right]^{1/2}}$$

$$= \frac{1}{2}\left(\frac{1}{r_x}+\frac{1}{r_y}\right)+\frac{1}{2}\frac{\left(\frac{1}{r_x}-\frac{1}{r_y}\right)^2+4\frac{1}{r_{xy}^2}}{\left[\left(\frac{1}{r_x}-\frac{1}{r_y}\right)^2+\frac{4}{r_{xy}^2}\right]^{1/2}}$$

$$= \frac{1}{2}\left(\frac{1}{r_x}+\frac{1}{r_y}\right)+\frac{1}{2}\left[\left(\frac{1}{r_x}-\frac{1}{r_y}\right)^2+\frac{4}{r_{xy}^2}\right]^{1/2} \qquad \text{...eqn. 2.15}(a)$$

If we take (Ref. Fig. 2.4)

$$\cos 2\alpha_2 = \frac{-\left(\frac{1}{r_x}-\frac{1}{r_y}\right)}{\left[\left(\frac{1}{r_x}-\frac{1}{r_y}\right)^2+\frac{4}{r_{xy}^2}\right]^{1/2}}$$

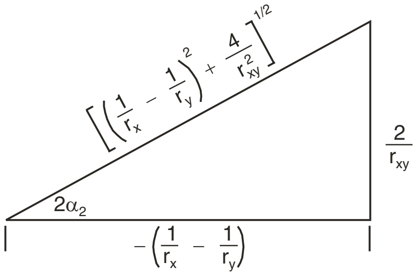


Fig. 2.4

and

$$\sin 2\alpha_2 = \frac{2/r_{xy}}{\left[\left(\frac{1}{r_x}-\frac{1}{r_y}\right)^2+\frac{4}{r_{xy}^2}\right]^{1/2}}$$

we get,

$$\frac{1}{r_2} = \frac{1}{2}\left(\frac{1}{r_x}-\frac{1}{r_y}\right)-\frac{1}{2}\left[\left(\frac{1}{r_x}-\frac{1}{r_y}\right)^2+\frac{4}{r_{xy}^2}\right]^{1/2} \qquad \text{...eqn. 2.15}(b)$$

It may be noted that $2\alpha_1$ and $2\alpha_2$ differ by 180° *i.e.* α_1 and α_2 differ by 90°.

The equations for principal curvatures are similar to those obtained for finding principal stresses. Hence Mohr's circle can be used to determine principal curvatures also. Figure 2.5 shows Mohr's circle for principal curvatures.

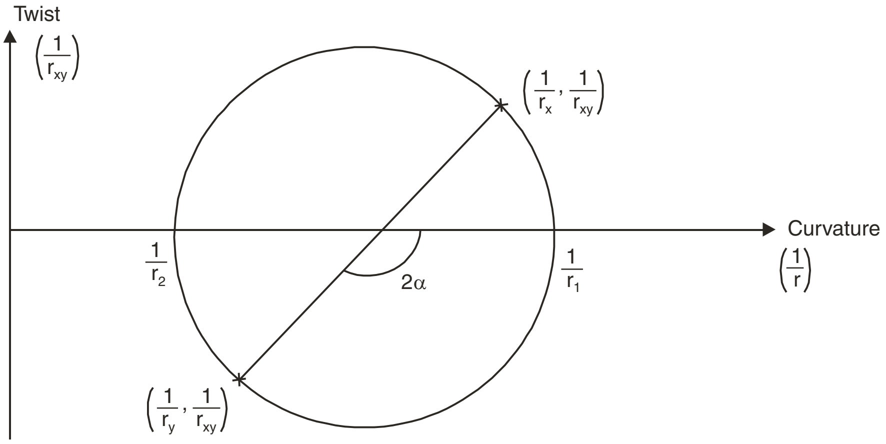


Fig. 2.5 Mohr's circle for curvatures

2.4 DISPLACEMENT—STRAIN RELATIONS

Let,

u — displacement in *x*-direction
v — displacement in *y*-direction and
w — displacement in *z*-direction.

u,v and *w* are considered positive when they are in positive directions of the coordinates *x, y* and *z.*

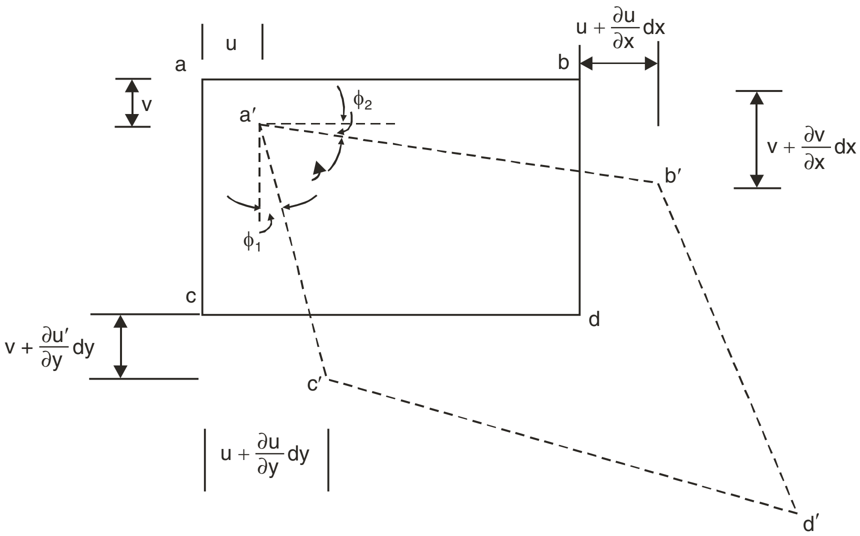


Fig. 2.6 Deflected shape of element

Consider an element of size $dx \times dy$ as shown in Fig. 2.6. Note that the original element *abcd* is deflected to the position $a'b'c'd'$. In the figure $a'b'c'd'$ is shown such that the deflections increase in the increasing directions of coordinates. Thus

(u, v) are deflections at a'.

$$\left(u + \frac{\partial u}{\partial x}dx,\ v + \frac{\partial v}{\partial x}dx\right) \text{ are deflections at } b'.$$

$$\left(u + \frac{\partial u}{\partial y}dy,\ v + \frac{\partial v}{\partial y}dy\right) \text{ are deflections at } c'.$$

∴ Change in length of the element in x-direction $= u + \frac{\partial u}{\partial x}dx - u = \frac{\partial u}{\partial x}dx.$

∴ Strain in x-direction

$$\varepsilon_x = \frac{\text{Change in length in } x\text{-direction}}{\text{Original length in } x\text{-direction}}$$

$$= \frac{\frac{\partial u}{\partial x}dx}{dx} = \frac{\partial u}{\partial x}$$

Similarly, strain in y-direction

$$\varepsilon_y = \frac{v + \frac{\partial v}{\partial y}\cdot dy - v}{dy} = \frac{\partial v}{\partial y}$$

Shearing strain

$$\gamma_{xy} = \phi_1 + \phi_2$$

$$= \frac{\frac{\partial v}{\partial x}\cdot dx}{dx} + \frac{\frac{\partial u}{\partial y}\cdot dy}{dy}$$

$$= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

Thus strains are given by the expressions

$$\varepsilon_x = \frac{\partial u}{\partial x}$$

$$\varepsilon_y = \frac{\partial v}{\partial y}$$

and $$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \qquad \text{...eqn. 2.16}$$

2.5 STRAINS IN TERMS OF *w*

Let a point at distance z from middle surface be displaced as shown in Fig. 2.7.

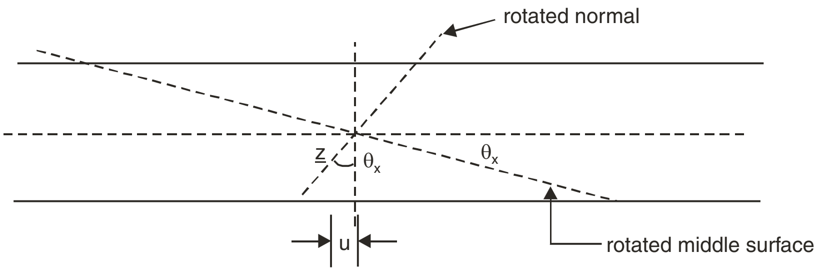


Fig. 2.7 Displacement u in terms of w

From the figure, it is clear that

$$u = -z\theta_x = -z\frac{\partial w}{\partial x}$$

Similarly,

$$v = -z\theta_y = -z\frac{\partial w}{\partial y}$$

∴

$$\varepsilon_x = \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = -z\frac{\partial^2 w}{\partial y^2}$$

and

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$= -z\frac{\partial^2 w}{\partial x \partial y} - z\frac{\partial^2 w}{\partial x \partial y}$$

$$= -2z\frac{\partial^2 w}{\partial x \partial y}$$

Thus,

$$\varepsilon_x = -z\frac{\partial^2 w}{\partial x^2}$$

$$\varepsilon_y = -z\frac{\partial^2 w}{\partial y^2}$$

and $$\gamma_{xy} = -2z\frac{\partial^2 w}{\partial x \partial y}$$ ...eqn. 2.17

The above expressions for strains ε_x and ε_y may be derived considering the curvatures also.

Referring to Fig. 2.8,

$$\varepsilon_x = \frac{\text{Final length} - \text{Original length}}{\text{Original length}}$$

$$= \frac{(r_x + z)\theta - r_x\theta}{r_x\theta}$$

$$= \frac{z}{r_x} = -z\frac{\partial^2 w}{\partial x^2}$$

Similarly $$\varepsilon_y = -z\frac{\partial^2 w}{\partial y^2}.$$

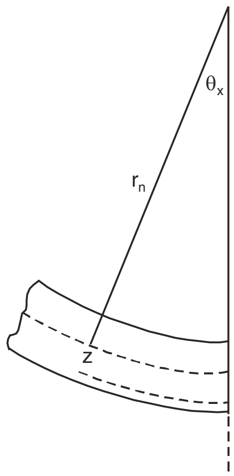


Fig. 2.8 Strain ε_x from curvature consideration

2.6 STRAIN-STRESS RELATIONS

Consider the element shown in Fig. 2.9, which is subjected to stresses σ_x, σ_y and τ_{xy}. In the figure stresses are shown in their positive senses.

Taking moment equilibrium condition about z-axis passing through A, we get

$\tau_{xy}\, dy\, h\, dx - \tau_{yx}\, h\, dx\, dy = 0$, where h is thickness of plate.

$\therefore$ $$\tau_{xy} = \tau_{yx}$$ ...(1)

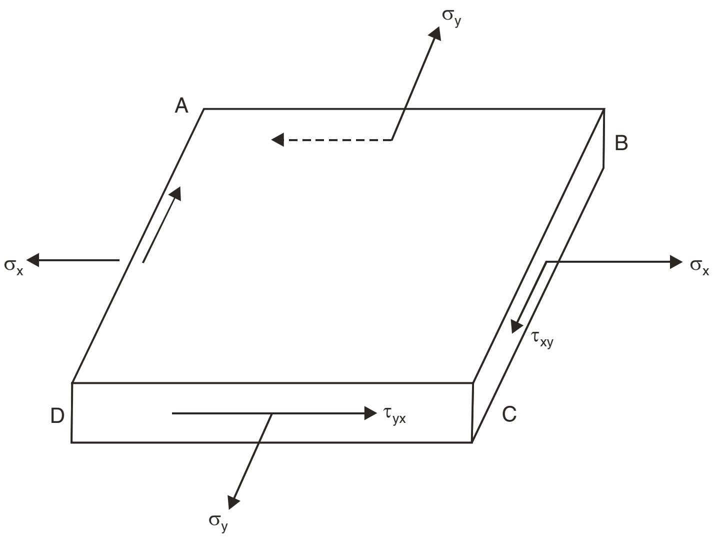


Fig. 2.9 Stresses on an element

From theory of elasticity, we know

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \mu\sigma_y\right) \qquad ...(2)$$

and

$$\varepsilon_y = \frac{1}{E}\left(\sigma_y - \mu\sigma_x\right) \qquad ...(3)$$

where E - Young's modulus
and μ - Poisson's ratio.

From eqn. 3,

$$\mu\varepsilon_y = \frac{1}{E}\mu\left(\sigma_y - \mu\sigma_x\right) \qquad ...(4)$$

Adding eqns. (2) and (4), we get

$$\varepsilon_x + \mu\varepsilon_y = \frac{1}{E}\left(1-\mu^2\right)\sigma_x$$

or

$$\sigma_x = \frac{E}{1-\mu^2}\left(\varepsilon_x + \mu\varepsilon_y\right) = -\frac{Ez}{1-\mu^2}\left(\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right)$$

Similarly,

$$\sigma_y = -\frac{Ez}{1-\mu^2}\left(\mu\frac{\partial^2 w}{\partial \lambda^2} + \frac{\partial^2 w}{\partial y^2}\right)$$

$$\tau_{xy} = G\gamma_{xy}$$

where G = Modulus of rigidity $= \dfrac{E}{2(1+\mu)}$

$\therefore$

$$\tau_{xy} = \frac{E}{2(1+\mu)}\gamma_{xy}$$

$$= -\frac{E}{2(1+\mu)}2z\frac{\partial^2 w}{\partial x \partial y}$$

$$= -\frac{Ez}{(1+\mu)}\cdot\frac{\partial^2 w}{\partial x \partial y}$$

$$= -\frac{Ez(1-\mu)}{1-\mu^2}\frac{\partial^2 w}{\partial x \partial y}$$

Thus,

$$\sigma_x = -\frac{Ez}{1-\mu^2}\left(\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right)$$

$$\sigma_y = -\frac{Ez}{1-\mu^2}\left(\mu\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) \qquad ...\text{eqn. 2.18}$$

and

$$\tau_{xy} = -\frac{Ez}{1-\mu^2}(1-\mu)\frac{\partial^2 w}{\partial x \partial y}$$

2.7 EXPRESSIONS FOR MOMENTS

Let,

M_x = Moment per unit length acting in x-direction.

M_y = Moment per unit length acting in y-direction.

and M_{xy} = Twisting moment per unit length w.r.t. $x - y$ directions.

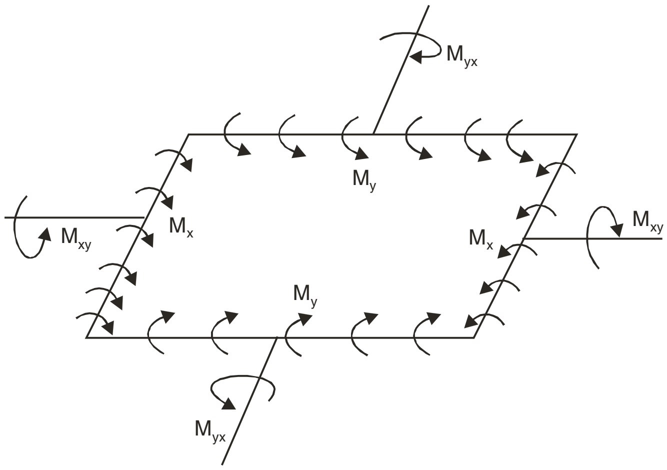
Sign convention used: Positive forces in positive side of z-coordinate produce +ve moments.

It amounts to taking sagging bending moments as positive moments. These moments may be represented by any one of the way shown in Fig. 2.10.

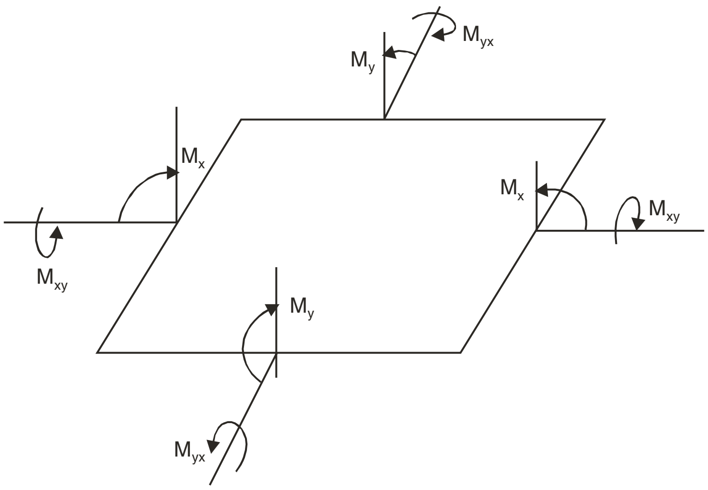
Note carefully:

M_{xy} is moment per unit length on face x in y-direction.

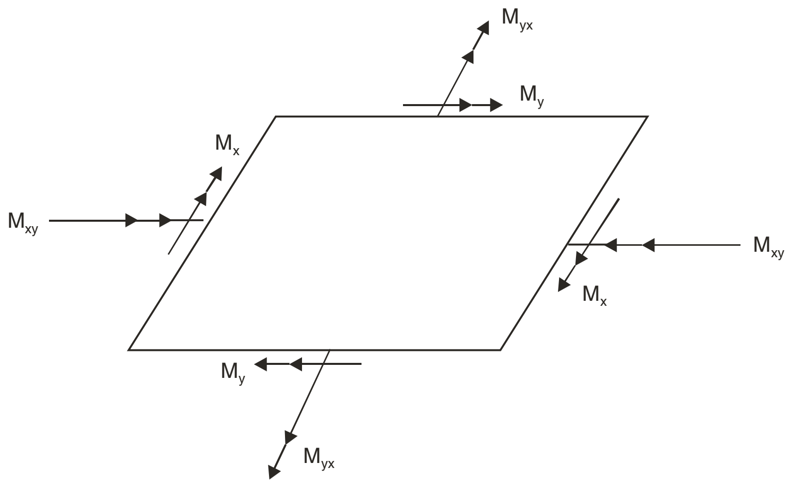
M_{yx} is moment per unit length on face y in x-direction.



(a)



(b)



(c)

Fig. 2.10 Sign convention for moments

The twisting moments are positive when they are as shown in Fig. 2.10 (Positive shear acting in positive direction of z produces positive twisting moment).

Now, $$M_x = \int_{-h/2}^{h/2} \sigma_z z \times 1 \times dz$$

$$= \int_{-h/2}^{h/2} -\frac{Ez}{1-\mu^2}\left(\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right) z\,dz$$

$$= -\frac{E}{1-\mu^2}\left(\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right)\left[\frac{z^3}{3}\right]_{-h/2}^{h/2}$$

$$= -\frac{Eh^3}{12\left(1-\mu^2\right)}\left(\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right)$$

$$= -D\left(\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right)$$

where $D = \frac{Eh^3}{12\left(1-\mu^2\right)}$ is flexural rigidity of plate.

Similarly $$M_y = -D\left(\mu\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)$$

Now,
$$M_{xy} = \int_{-h/2}^{h/2} \tau_{xy} z \times 1 \times dz$$

$$= \int_{-h/2}^{h/2} \frac{-Ez(1-\mu)}{1-\mu^2} \frac{\partial^2 w}{\partial x \partial y} z\,dz$$

$$= -\frac{Eh^3(1-\mu)}{12(1-\mu^2)} \frac{\partial^2 w}{\partial x \partial y}$$

$$= -D(1-\mu)\frac{\partial^2 w}{\partial x \partial y}.$$

Thus,

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2}\right)$$

$$M_y = -D\left(\mu \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) \qquad \text{...eqn. 2.19}$$

$$M_{xy} = -D(1-\mu)\frac{\partial^2 w}{\partial x \partial y}.$$

2.8 MOMENT IN ANY DIRECTION

Let '*n*' be the direction making angle α in clockwise direction to *x*-direction as shown in Fig. 2.11. Consider the element of size $dx \times dy$ and thickness *h*. Now we have to find expression for moment in *n*-direction in terms of known moments M_x, M_y and M_{xy}.

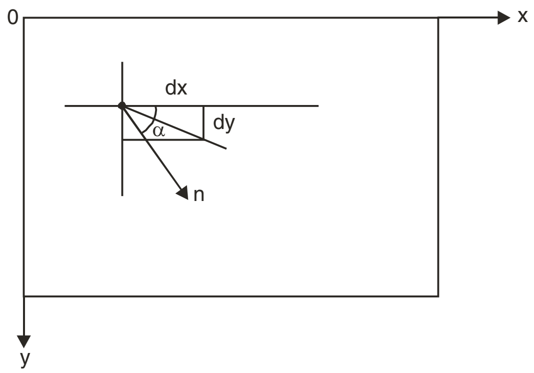


Fig. 2.11 Element considered

Now consider the stresses acting on the triangular element as shown in Fig. 2.12.

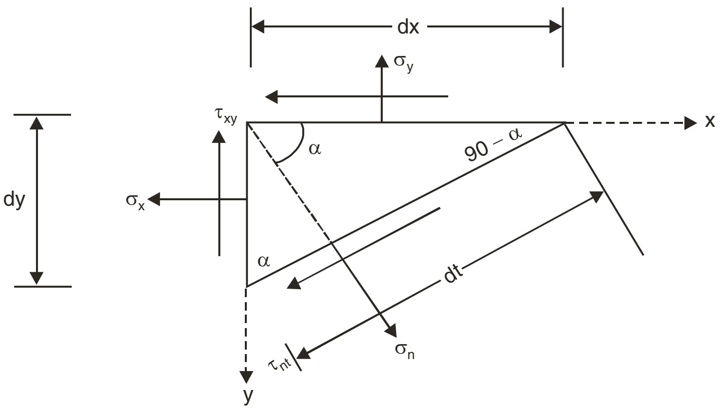


Fig. 2.12 Triangular element with stresses on its sides

Noting that $\tau_{xy} = \tau_{yx}$,

$$\sum \text{Forces in } n \text{ direction} = 0, \text{ gives}$$

$\sigma_n h\,dt = \sigma_x h\,dy \cos\alpha + \sigma_y h\,dx \sin\alpha + \tau_{xy} h\,dy \sin\alpha + \tau_{xy} h\,dx \cos\alpha$

Throughout dividing by $h.dt$ and noting that

$$\frac{dx}{dt} = \sin\alpha \text{ and } \frac{dy}{dt} = \cos\alpha, \text{ we get}$$

$$\sigma_n = \sigma_x \cdot \cos^2\alpha + \sigma_y \sin^2\alpha + \tau_{xy}\cos\alpha\sin\alpha + \tau_{xy}\sin\alpha\cdot\cos\alpha$$

$$= \sigma_x \cos^2\alpha + \sigma_y \sin^2\alpha + 2\tau_{xy}\sin\alpha\cdot\cos\alpha$$

$$= \sigma_x\left(\frac{1+\cos 2\alpha}{2}\right) + \sigma_y\left(\frac{1-\cos 2\alpha}{2}\right) + \tau_{xy}\sin 2\alpha$$

$$= \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos 2\alpha + \tau_{xy}\sin 2\alpha$$

From equation of equilibrium

$$\Sigma F_t = 0, \text{ we get}$$

$$\tau_{nt} h dt = -\sigma_x h dy \sin\alpha + \sigma_y h dx \cos\alpha$$

$$-\tau_{xy} h dx \sin\alpha + \tau_{xy} h dy \cos\alpha$$

$$\therefore \quad \tau_{nt} = -\sigma_x \cos\alpha\sin\alpha + \sigma_y \sin\alpha\cdot\cos\alpha + \tau_{xy}\left(\cos^2\alpha - \sin^2\alpha\right)$$

$$= \frac{-\sigma_x+\sigma_y}{2}\sin 2\alpha + \tau_{xy}\cos 2\alpha$$

$$\therefore \qquad M_n = \int_{-h/2}^{h/2} \sigma_n \times 1 \times dz \times z$$

$$= \int_{-h/2}^{h/2} \left(\frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\alpha + \tau_{xy} \sin 2\alpha \right) z\, dz$$

$$= \frac{M_x + M_y}{2} + \frac{M_x - M_y}{2} \cos 2\alpha + M_{xy} \sin 2\alpha \qquad \text{...eqn. 2.20}$$

Similarly,

$$M_{nt} = \int_{-h/2}^{h/2} \tau_{xy} \times 1 \times dz \times z$$

$$= \frac{-M_x + M_y}{2} \sin 2\alpha + M_{xy} \cos 2\alpha \qquad \text{...eqn. 2.21}$$

2.9 PRINCIPAL MOMENTS

The planes on which twisting moment is zero are known as principal planes of moment. From eqn. 2.21, if α_1 is the direction of principal planes, we get

$$0 = \frac{-M_x + M_y}{2} \sin 2\alpha_1 + M_{xy} \cos 2\alpha_1$$

i.e.

$$\tan 2\alpha_1 = \frac{2M_{xy}}{M_x - M_y} \qquad \text{...eqn. 2.22}$$

It can be easily proved that moments on principal planes have extreme values. For moment M_n to have extreme value, necessary condition is,

$$\left. \frac{\partial M_n}{d\alpha} \right|_{\alpha=\alpha'} = 0$$

i.e.

$$\frac{M_x - M_y}{2} (-2 \sin 2\alpha') + M_{xy} 2 \cos 2\alpha' = 0$$

i.e.

$$\tan 2\alpha' = \frac{2M_{xy}}{M_x - M_y} \quad \text{which is similar to eqn. 2.22.}$$

Thus $\alpha_1 = \alpha'$ *i.e.* the moments on principal planes are extreme values. It can be shown that the magnitude of principal moments are

$$M_{1,2} = \frac{M_x + M_y}{2} \pm \left[\left(\frac{M_x - M_y}{2} \right)^2 + M_{xy}^2 \right]^{1/2}$$

The expressions are similar to the expressions for principal stresses. Hence for moments also Mohr's circle can be drawn. Figure 2.13 shows Mohr's circle for moments.

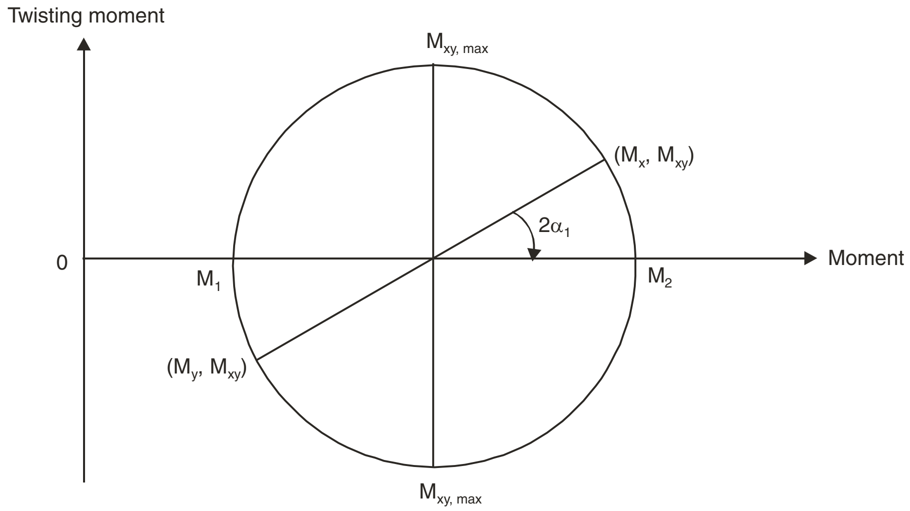


Fig. 2.13 Mohr's circle for moments

QUESTIONS

1. Prove that in a slightly bent plate under pure bending, the directions of maximum slope and zero slope are at right angles to each other. Find the expression for maximum slope.
2. Prove that the sum of curvatures in any two mutually perpendicular directions in a slightly bent plate is constant.
3. Derive the expression for curvature in a slightly bent plate in any direction under pure moment and then determine the expressions for
   (*a*) principal curvature direction.
   (*b*) values of principal curvatures.
4. Show that planes of principal curvatures are the planes of extreme curvatures also.
5. Derive the expressions for strains in a plate in terms of single displacement '*w*'.
6. State the strain-stress relations and strain-displacement relations in terms of single displacement '*w*'. Hence establish the expressions for stresses and moments in terms of '*w*'.
7. Derive the expression for moment in any direction and then determine:
   (*a*) Principal planes for moment.
   (*b*) Value of principal moments.
8. At a point in a plate, the moments are as shown below:
   M_x = 90 kn-m M_y = 50 kn-m M_{xy} = 30 kn-m
   Determine
   (*a*) Direction of principal planes.
   (*b*) Maximum/Minimum moments.
   (*c*) Maximum twisting moment and its direction.

Chapter

3

Small Deflections of Laterally Loaded Plates

Plates are usually subjected to lateral loads and bend in both directions. The bending moment and shear forces vary from point to point. Hence, the moments and shear forces on negative face and positive face of an element will not be same. However, the element will be in equilibrium under the action of these stress resultants and the load on it. In this chapter, the equilibrium equation is derived and the boundary conditions to be considered are discussed.

3.1 STRESS RESULTANTS ON A TYPICAL PLATE ELEMENT

Figure 3.1 shows a typical element and the stress resultant on it. Stress resultants on negative faces (at x and y sections) are noted without superscripts while the stress resultants on positive faces (at $x + dx$ and $y + dy$) are shown with '+' sign as superscript.

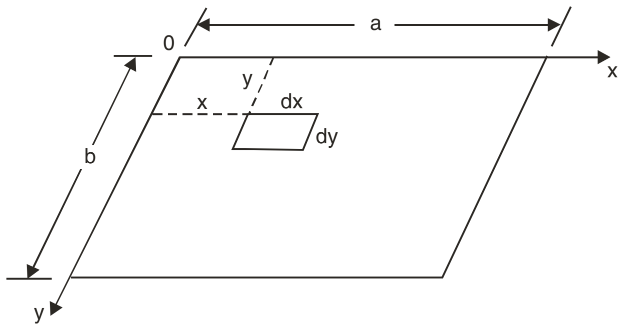
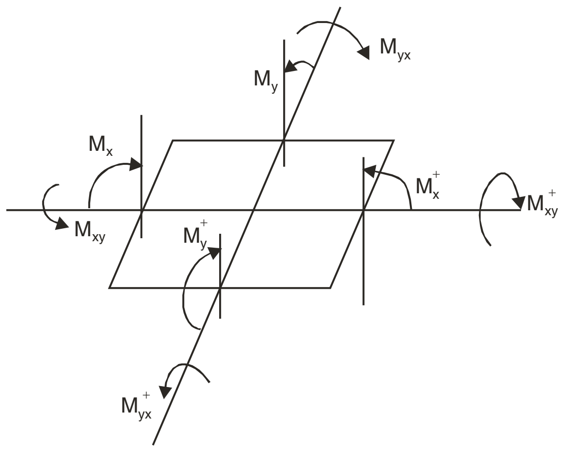


Fig. 3.1 (*a*) Position of element

Consider the moments in x-direction.
on negative face moment = M_x
on positive face moment = M_x^+

If the rate of change of moment in x-direction is $\frac{\partial M_x}{\partial x}$, then

$$M_x^+ = M_x + \frac{\partial M_x}{\partial x} dx$$



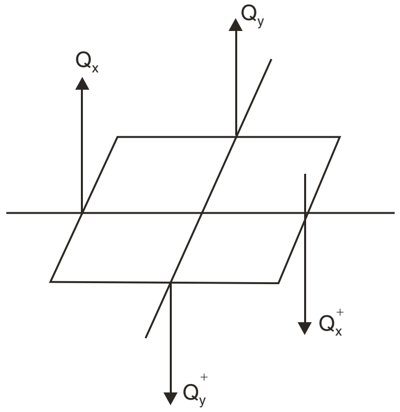


Fig. 3.1 (*b*) Moments on element (*c*) Vertical shears on element

Similarly the relations of stress resultants on positive faces with negative faces may be written. Thus,

$$\left.\begin{array}{ll} M_x^+ = M_x + \dfrac{\partial M_x}{\partial x} dx & M_y^+ = M_y + \dfrac{\partial M_y}{\partial y} dy \\ M_{xy}^+ = M_{xy} + \dfrac{\partial M_{xy}}{\partial x} dx & M_{yx}^+ = M_{yx} + \dfrac{\partial M_{yx}}{\partial y} dy \\ Q_x^+ = Q_x + \dfrac{\partial Q_x}{\partial x} dx & Q_y^+ = Q_y + \dfrac{\partial Q_y}{\partial y} dy \end{array}\right\} \quad \text{...eqn. 3.1}$$

Note that all stress resultants are per unit length.

Apart from these stress resultants, load of intensity q per unit area is acting on the element in the downward direction. Hence, total downward load on the element.

$$= q\ dx\ dy \quad \text{...eqn. 3.2}$$

3.2 EQUATIONS OF EQUILIBRIUM

Three independent equations of equilibrium can be written for the element—one considering the forces in z-direction and two moment equilibrium for the moments in x-direction and y-direction

Consider $\Sigma F_z = 0$. It gives

$$Q_x^+\ dy - Q_x\ dy + Q_y^+\ dx - Q_y\ dy + q\ dx\ dy = 0$$

i.e.

$$\left(Q_x + \frac{\partial Q_x}{\partial x} dx\right) dy - Q_x dy + \left(Q_y + \frac{\partial Q_y}{\partial y} dy\right) dx - Q_y dy + q\ dx\ dy = 0$$

Simplifying and then dividing throughout by $dx\ dy$, we get

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 0 \quad \text{...eqn. 3.3}$$

Equilibrium equation,

$$\Sigma M_x = 0, \text{ gives}$$

$$M_x^+ dy - M_x dy + M_{yx}^+ dx - M_{yx} dx - Q_x^+ dy \frac{dx}{2} - Q_x dy \frac{dx}{2} = 0$$

$$\left(M_x + \frac{\partial M_x}{\partial x} dx \right) dy - M_x dy + \left(M_{yx} + \frac{\partial M_{yx}}{\partial y} dy \right) dx - M_{yx} dx$$

$$- \left(Q_x + \frac{\partial Q_x}{\partial x} dx \right) dy \cdot \frac{dx}{2} - Q_x dy \frac{dx}{2} = 0$$

Simplifying and neglecting small quantity of higher order, we get

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{yx}}{\partial y} - Q_x = 0$$

i.e.
$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{yx}}{\partial y} = Q_x \qquad \text{...eqn. 3.4}$$

Similarly moment equilibrium in y-direction, gives

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} = Q_y \qquad \text{...eqn. 3.5}$$

Substituting the values of Q_x and Q_y as shown in Eqns. 3.4 and 3.5 in equation 3.3, we get

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_{yx}}{\partial x \partial y} + \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + q = 0$$

i.e.
$$\frac{\partial^2 M_x}{\partial x} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -q$$

But we know (Refer eqn. 2.20)

$$M_x = -D \left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2} \right)$$

$$M_y = -D \left(\mu \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)$$

and
$$M_{xy} = -D(1 - \mu) \frac{\partial^2 w}{\partial x \partial y}$$

Hence, equation of equilibrium is

$$-D \left(\frac{\partial^4 w}{\partial x^4} + \mu \frac{\partial^4 w}{\partial x^2 \partial y^2} \right) - 2D(1 - \mu) \frac{\partial^4 w}{\partial x^2 \partial y^2} - D \left(\mu \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^2} \right) = -q$$

i.e.
$$-D \left(\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) = -q.$$

i.e.
$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}$$

i.e.
$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = \frac{q}{D}$$

Denoting $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ by ∇ we get

$$\nabla^2\left(\nabla^2 w\right) = \frac{q}{D}$$

or
$$\nabla^4 w = \frac{q}{D} \qquad \text{...eqn. 3.6}$$

Equation 3.6 is known as *Equation of Plates* or *Lagrange Equation for Plates.*

3.3 EXPRESSIONS FOR VERTICAL SHEARS

From equation 3.4,

$$Q_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y}$$

But
$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right)$$

and
$$M_{xy} = -D(1-\mu)\frac{\partial^2 w}{\partial x \partial y}$$

$\therefore$
$$Q_x = -D\left(\frac{\partial^3 w}{\partial x^3} + \mu\frac{\partial^3 w}{\partial x \partial y^2}\right) - D(1-\mu)\frac{\partial^3 w}{\partial x \partial y^2}$$

$$= -D\left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2}\right)$$

$$= -D\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right)$$

$$= -D\frac{\partial}{\partial x}\left(\nabla^2 w\right)$$

Similarly

$$Q_y = -D\frac{\partial}{\partial y}\left(\nabla^2 w\right)$$

Thus $$Q_x = -D\frac{\partial}{\partial x}\left(\nabla^2 w\right)$$

$$Q_y = -D\frac{\partial}{\partial y}\left(\nabla^2 w\right) \qquad \text{...eqn. 3.7}$$

3.4 BOUNDARY CONDITIONS

(*a*) **Fixed Edge: If the edge *x* = *a* is fixed** (Ref. Fig. 3.2)

$$w = 0 \text{ and } \frac{\partial w}{\partial x} = 0 \qquad \text{...eqn. 3.8}$$

(*b*) **If edge *x* = *a* is simply supported.**

$$w|_{x=a} = 0$$

and $$M_x|_{x=a} = 0$$

i.e. $$-D\left(\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right) = 0$$

But along $x = a$, $\frac{\partial^2 w}{\partial y^2} = 0,$ since the edge is supported throughout.

Hence, the boundary condition is

$$\left.\frac{\partial^2 w}{\partial x^2}\right|_{x=a} = 0.$$

Thus, the boundary conditions are

$$w|_{x=a} = 0 \text{ and } \frac{\partial^2 w}{\partial x^2} = 0 \qquad \text{...eqn. 3.9}$$

(*c*) **Free edge at *x* = *a***

If the edge $x = a$ is free,

$$M_x = 0,\ M_{xy} = 0 \text{ and } Q_x = 0.$$

The above three boundary conditions were expressed by Poisson. But Kelvin felt that there is something wrong, since when at all other edges two conditions are found, how there can be three edge conditions in this case.

Kelvin and Tait pointed out that the last two conditions are not independent. They can be combined to give a single realistic condition.

Referring to Figure 3.2, twisting moment on an elemental length *dy* is $M_{xy}\,dy$. This moment may be replaced by two vertical shears of magnitude M_{xy} separated by *dy*. This change will not alter the behaviour of the plate. Kelvin and Tail pointed out that the actual boundary condition is at any point on free edge, vertical shear plus vertical shear due to replacement of twisting moment must be zero. Now, vertical force at any point due to M_{xy}

$$= M_{xy}^{+} - M_{xy}$$

$$= M_{xy} + \frac{\partial M_{xy}}{\partial y} dy - M_{xy} = \frac{\partial M_{xy}}{\partial y} dy, \text{ downward.}$$

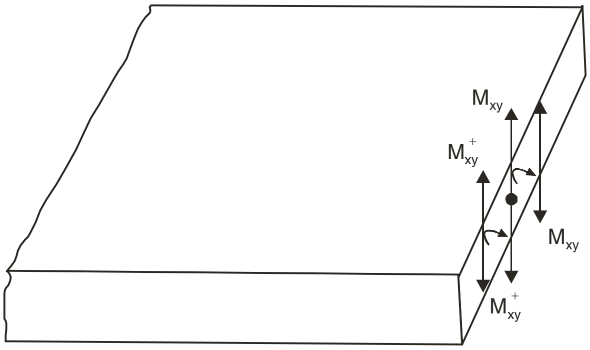


Fig. 3.2 Contribution of twisting moment to vertical shear

∴ Total vertical shear

$$V_x dy = Q_x dy + \frac{\partial M_{xy}}{\partial y} dy$$

∴ $$V_x = Q_x + \frac{\partial M_{xy}}{\partial y}$$

$$= -D\left[\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)\right] - D(1-\mu)\frac{\partial^3 w}{\partial x \partial y^2}$$

$$= -D\left[\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2}\right] - D(1-\mu)\frac{\partial^3 w}{\partial x \partial y^2}$$

$$= -D\left[\frac{\partial^3 w}{\partial x^3} + (2-\mu)\frac{\partial^3 w}{\partial x \partial y^2}\right]$$

The real boundary condition is

$$V_x = 0$$

Thus, at free edge the boundary conditions are, $M_x\big|_{x=a} = 0$ and $V_x\big|_{x=a} = 0$ ...eqn. 3.10

(*d*) **If edge *x* = *a* is elastically supported.**

Figure 3.4 shows this case, in which edge $x = a$ is supported elastically by a beam. Let flexural rigidity of beam be B and torsional rigidity be G.

We know, for the beam,

$$B\frac{\partial^4 w}{\partial y^4} = \text{Load intensity}$$

One boundary condition is,

deflection of beam = deflection of plate

$$B\left.\frac{\partial^4 w}{\partial y^4}\right|_{x=a} = V\big|_{x=a}$$

$$= -D\left[\frac{\partial^3 w}{\partial x^3} + (2-\mu)\frac{\partial^3 w}{\partial x \partial y^2}\right]_{x=a}$$

The second boundary condition may be written by considering the torsional rotation of elemental length of beam. Referring to Figure 3.3,

Let M_{Tb} be twisting moment in the beam. Twisting of the beam is due to moment $M_x\big|_{x=a}$ in the plate. From moment equilibrium condition, we get,

$$M_{Tb}^{+}dy - M_{Tb}dy = M_x dy$$

i.e.
$$M_{Tb}\,dy + \frac{\partial M_{Tb}}{\partial y}dy + M_{Tb}\,dy = Mxdy$$

$$\frac{\partial M_{Tb}}{\partial y} = -M_x$$

$$\frac{\partial}{\partial y}\left(G\cdot\frac{\partial^2 w}{\partial x \partial y}\right) = D\left(\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right),$$

since w is same for both beam and plate.

i.e.
$$G\frac{\partial^3 w}{\partial x \partial y} = D\left(\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right)$$

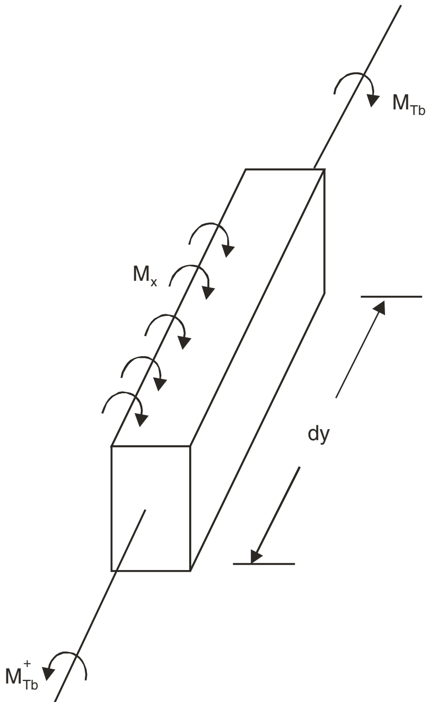


Fig. 3.3 Torsional rotation of edge beam at $x = a$

Thus, for plate with elastic support on beam, the boundary conditions are,

$$B\frac{\partial^4 w}{\partial y^4} = \frac{\partial^3 w}{\partial x^3} + (2-\mu)\frac{\partial^3 w}{\partial x \partial y^2}$$

...eqn. 3.11

and
$$G\frac{\partial^3 w}{\partial x \partial y^2} = D\left(\frac{\partial^3 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right)$$

3.5 CORNER REACTION

Consider the corner $x = a$ and $y = b$ in which the edges $x = a$ and $y = b$ are discontinuous. If we resort to replacing twisting moment in the last element along the edge $x = a,$ we find there is an upward force of $\frac{\partial M_{xy}}{\partial y}$, corner. Similarly if we replace twisting moment in the last element along $y = b,$ we find there is an upward force of $\frac{\partial M_{yx}}{\partial x}$, corner. Thus, at corner $x = a$, $y = b$, there is net upward force of $\frac{\partial M_{xy}}{\partial y}$ corner plus $\frac{\partial M_{yx}}{\partial x}$ corner. Since $M_{xy} = M_{yx}$, we can conclude at the corner there is net upward force of $2\frac{\partial M_{xy}}{\partial x}$. Hence, at discontinuous corners, lifting takes place. To take care of this phenomenon, in R.C.C. slabs corner reinforcements are provided.

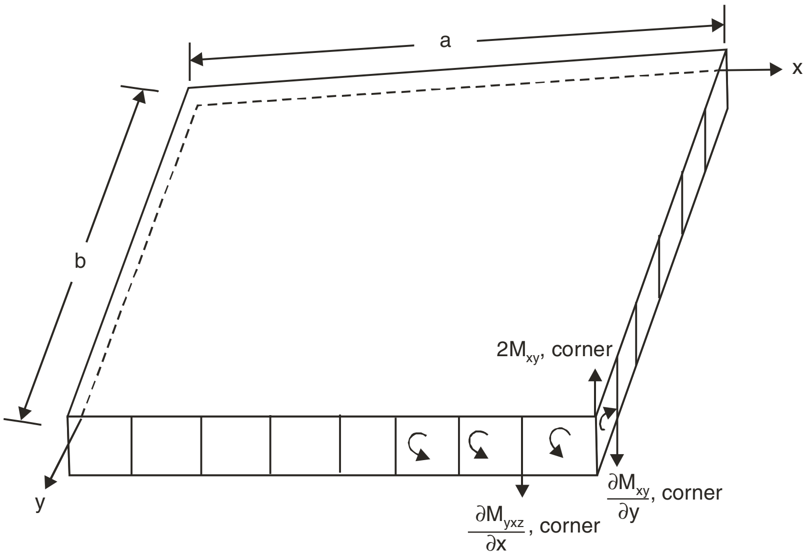


Fig. 3.4 Corner reaction

QUESTIONS

1. Derive the biharmonic equation for slightly bent thin plate.
2. Discuss the boundary conditions at free edge.
3. Explain why there is tendency of uplift at corner if two adjacent edges of a plate are discontinuous.

Chapter

4

Fourier Series of Loadings

It is difficult to solve the biharmonic equation of the plate, if the loading is considered as it is. However satisfactory solution can be obtained easily if loading is expressed in a series of equivalent sinusoidal loading form. Such loading is called Fourier series loading. Though the loading is expressed in the infinite series, first term itself gives 95 percentage of deflection and 90 percentage of moment. Hence, Fourier series solutions are used in the analysis of plates and shells.

4.1 FOURIER SERIES OF LOADS ON BEAMS

A given loading is considered as a sum of series of sinusoidal loadings. The peak values of sinusoidal loadings are so selected that if a number of such terms are added original loading is obtained. The Fourier equivalent terms for a given loading are given by,

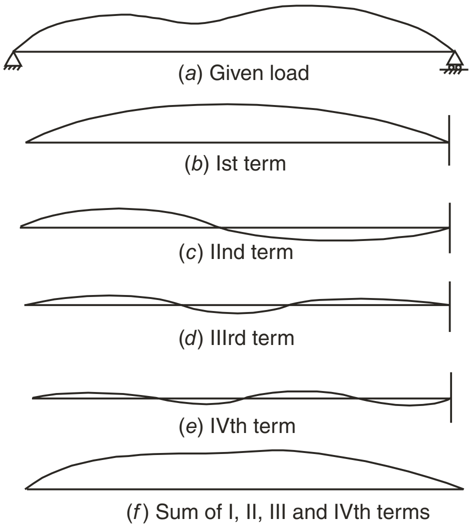


Fig. 4.1

$$f(x) = \sum_{1}^{\infty} a_m \sin \frac{m\pi x}{L} \quad \text{where}$$

$$a_m = \frac{2}{L} \int f(x) \sin \frac{m\pi x}{L} dx. \qquad \text{...eqn. 4.1}$$

where L is span of the beam.

Figure 4.1 shows the addition of such loadings to get original loading $f(x)$.

To get reasonably good loading, it may be necessary to take 20–25 terms in Fourier series. But deflections can be obtained reasonably well (within 5 percent) with the first term itself. Similarly bending moments can be obtained reasonably well with 3 terms. Hence, this approach has been successfully used for the analysis of plates.

4.2 FOURIER SERIES FOR UDL ON BEAMS

Expression for equivalent Fourier series is derived in this article for uniformly distributed load acting on a beam of span L (Refer Figure 4.2).

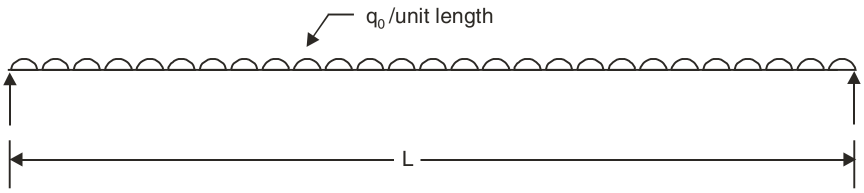


Fig. 4.2 UDL over entire span

In general for any loading $q(x)$, Fourier equivalent is

$$q(x) = \sum_{m=1}^{\infty} a_m \sin \frac{m\pi x}{L}$$

where
$$a_m = \frac{2}{L}\int_0^L q(x) \sin \frac{m\pi x}{L} dx \qquad \text{...eqn. 4.2}$$

In case of uniformly distributed load,

$$q(x) = q_0$$

$$\therefore \quad a_m = \frac{2}{L}\int_0^L q_0 \sin \frac{m\pi x}{L} dx$$

$$= \frac{2}{L}\frac{L}{m\pi}\left[-q_0 \cos \frac{m\pi x}{L}\right]_0^L$$

$$= \frac{2q_0}{m\pi}[-\cos m\pi + 1]$$

$\therefore$ For odd values of m, $a_m = \dfrac{4q_0}{m\pi}$ and for even values of m, $a_m = 0$

$$\boxed{\therefore \quad q(x) = \sum_{m=1,3,\ldots}^{\infty} \frac{4q_0}{m\pi} \sin \frac{m\pi x}{L}} \qquad \text{...eqn. (4.3)}$$

4.3 FOURIER EQUIVALENT FOR UDL OVER A SMALL LENGTH

Let load intensity be q over a length u with its centre of gravity at ξ acting on a beam of length L as shown in Figure 4.3. It is required to find Fourier equivalent load for this.

In general

$$q(x) = \sum_{m=1}^{\infty} a_m \sin \frac{m\pi x}{L}$$

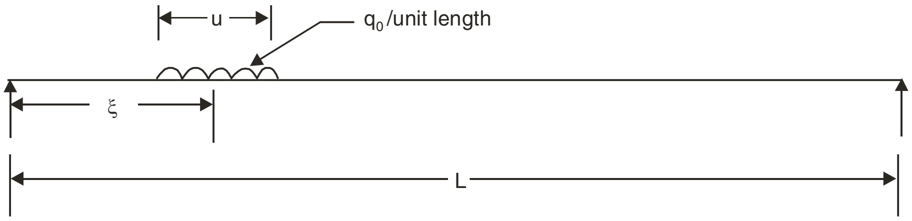


Fig. 4.3 UDL over a small length

where
$$a_m = \frac{2}{L}\int_0^L q(x) \sin \frac{m\pi x}{L} dx$$

In this case,

$$q(x) = 0 \quad \text{for} \quad x = 0 \text{ to } \xi - \frac{u}{2}$$

$$= q_0 \quad \text{for} \quad x = \xi - \frac{u}{2} \text{ to } \xi + \frac{u}{2}$$

$$= 0 \quad \text{for} \quad x = \xi + \frac{u}{2} \text{ to } L.$$

$\therefore$
$$a_m = \frac{2}{L}\left[\int_0^{\xi - u/2} 0\, dx + \int_{\xi - u/2}^{\xi + u/2} \sin \frac{m\pi x}{4} dx + \int_{\xi + u/2}^{L} 0\ dx\right]$$

$$= \frac{2}{L} \cdot q_0 \left[-\frac{L}{m\pi} \cos \frac{m\pi x}{L}\right]_{\xi - u/2}^{\xi + u/2}$$

$$= \frac{2}{L} q_0 \frac{L}{m\pi}\left[-\cos \frac{m\pi(\xi + u/2)}{L} + \cos \frac{m\pi(\xi - u/2)}{L}\right]$$

$$= \frac{2q_0}{m\pi} \cdot 2 \sin \frac{m\pi\xi}{L} \cdot \sin \frac{m\pi u}{2L}$$

$$= \frac{4q_0}{m\pi} \sin \frac{m\pi\xi}{L} \cdot \sin \frac{m\pi u}{2L}$$

$$q(x) = \sum_{m=1}^{\infty} \frac{4q_0}{m\pi} \sin\frac{m\pi\xi}{L} \sin\frac{m\pi u}{2L} \sin\frac{m\pi x}{L} \qquad \text{...eqn. 4.4}$$

4.4 FOURIER EQUIVALENT FOR A CONCENTRATED LOAD

Figure 4.4 shows a concentrated load P at distance ξ from left support acting on a beam of span L.

Fig. 4.4 Concentrated Load *P*

Its Fourier equivalent may be obtained from expression 4.4 by treating it as the case in which total load remains the same while u tends to zero. Thus, we have to substitute

$$P = q_0 u$$

and $$u \to 0.$$

As $u \to 0, \ \sin\frac{m\pi u}{2L} = \frac{m\pi u}{2L}$

$$\therefore \qquad q(u) = \sum_{m=1}^{\infty} \frac{4q_0}{m\pi} \sin\frac{m\pi\xi}{L} \frac{m\pi u}{2L} \sin\frac{m\pi x}{L}$$

i.e.
$$q(x) = \sum_{m=1}^{\infty} \frac{2P}{L} \sin\frac{m\pi\xi}{L} \cdot \sin\frac{m\pi x}{L} \qquad \text{...eqn. 4.7}$$

4.5 FOURIER EQUIVALENT FOR HYDROSTATIC LOAD

Figure 4.5 shows hydrostatic load varying from 0 intensity at $x = 0$ to q_0 at $x = L$

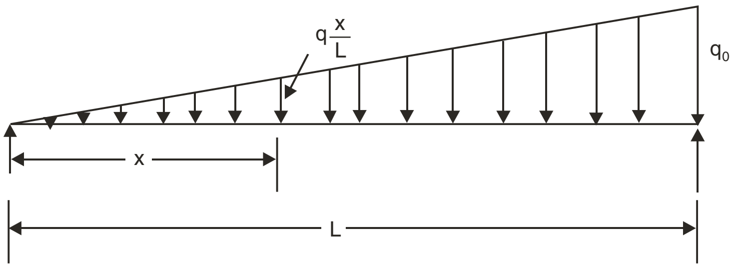


Fig. 4.5 Hydrostatic Load on a beam

Its Fourier equivalent is

$$q(x) = \sum_{m=1}^{\infty} a_m \sin\frac{m\pi x}{a}$$

where $$a_m = \frac{2}{L}\int_0^L q(x)\sin\frac{m\pi x}{a}\,dx$$

In this case $$q(x) = q_0\frac{x}{L}$$

$\therefore$ $$a_m = \frac{2}{L}\int_0^L \frac{q_0 x}{L}\sin\frac{m\pi x}{a}\,dx$$

$$= \frac{2q_0}{L^2}\int_0^L x\sin\frac{m\pi x}{a}\,dx$$

Noting that if u and v are the functions of x

$$\int uv\,dx = u\int v\,dx - \int\int v\frac{du}{dx}\,dx$$

and taking $u = x$ and $v = \sin\frac{m\pi x}{a}$, we get

$$a_m = \frac{2q_0}{L^2}\left[x\int\sin\frac{m\pi x}{L}dx - \int\int\sin\frac{m\pi x}{L}\times 1\times dx\right]$$

$$= \frac{2q_0}{L^2}\left[x\frac{L}{m\pi}\left(-\cos\frac{m\pi x}{L}\right) - \frac{L^2}{m^2\pi^2}\left(-\sin\frac{m\pi x}{L}\right)\right]_0^L$$

$$= \frac{2q_0}{L^2}\left[-\frac{L^2}{m\pi}\cos m\pi + \frac{L^2}{m^2\pi^2}(\sin m\pi)\right]$$

$$= \frac{2q_0}{m\pi}(-\cos m\pi),$$ since $\sin m\pi = 0$ for all values of m

$$= \frac{2q_0}{m\pi}(-1)^{m+1},$$ since for odd values, $-\cos m\pi = 1$

and for even values, $-\cos m\pi = -1$.

$\therefore$ Fourier equivalent for hydrostatic load is

$$q(x) = \sum_{m=1}^{\infty}\frac{2q_0}{m\pi}(-1)^{m+1}\sin\frac{m\pi x}{L}. \quad \text{...eqn. 4.8}$$

4.6 FOURIER EQUIVALENT FOR TRIANGULAR LOAD WITH PEAK VALUE AT MID-SPAN

This type of loading is shown in Fig. 4.6.

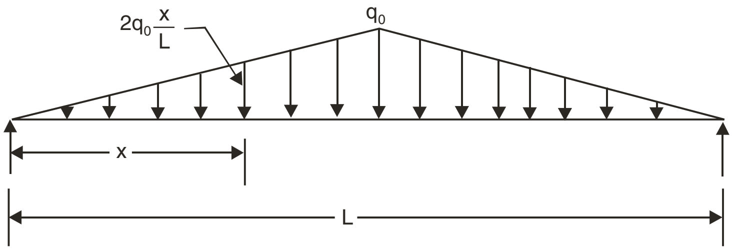


Fig. 4.6 Triangular load with peak value at mid span

The general form of Fourier expression is $q(x) = \sum a_m \sin\frac{m\pi x}{L}$

where $$a_m = \frac{2}{L}\int_0^L q(x)\sin\frac{m\pi x}{L}.$$

Now, for $$x = 0 \text{ to } L/2,$$

$$q(x) = \frac{2q_0 x}{L}$$

Due to symmetry, it is obvious that

$$\int_0^L q(x)\,dx = 2\int_0^{L/2} q(x)\,dx.$$

$\therefore$ $$a_m = \frac{2}{L}\times 2\int_0^{L/2}\frac{2q_0 x}{L}\sin\frac{m\pi x}{L}\,dx$$

$$= \frac{8q_0}{L^2}\int x\sin\frac{m\pi x}{L}\,dx.$$

Noting that $$\int x\sin\frac{m\pi x}{L}\,dx = x\int \sin\frac{m\pi x}{L}\,dx - \int\int \sin\frac{m\pi x}{L}\,dx$$

$$= \left[x\frac{L}{m\pi}\left(-\cos\frac{m\pi x}{L}\right) + \frac{L^2}{m^2\pi^2}\sin\frac{m\pi x}{L}\right]_0^{L/2}$$

$$= \left[-\frac{L^2}{2m\pi}\cos\frac{m\pi}{2} + \frac{L^2}{m^2\pi^2}\sin\frac{m\pi}{2}\right]$$

$$= \frac{L^2}{m^2\pi^2}\sin\frac{m\pi}{2}, \text{ since } \cos\frac{m\pi}{2} = 0 \quad \text{...eqn. 4.8}(a)$$

$$\therefore \qquad a_m = \frac{8q_0}{m^2\pi^2}\sin\frac{m\pi}{2}$$

$$\therefore \qquad \boxed{q(x) = \sum_{m=1}^{\infty}\frac{8q_0}{m^2\pi^2}\sin\frac{\pi}{2}\cdot\sin\frac{m\pi x}{L}} \qquad \text{...eqn. 4.8}(b)$$

4.7 DOUBLE FOURIER SERIES EXPRESSIONS FOR LOADS

The loads acting on a plate of size $a \times b$ (Refer Fig. 4.7) are to be expressed in Fourier series involving sine functions in x as well as in y directions.

It is required to find

$$q(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

Let $$g_m(y) = \sum_{n=1}^{\infty} a_{mn}\sin\frac{n\pi y}{b}$$

$$\therefore \qquad q(x, y) = \sum_{m=1}^{\infty} g_m(y)\sin\frac{m\pi x}{a}$$

$$\therefore \qquad g_m(y) = \frac{2}{a}\int_0^a q(x, y)\sin\frac{m\pi x}{a}dx$$

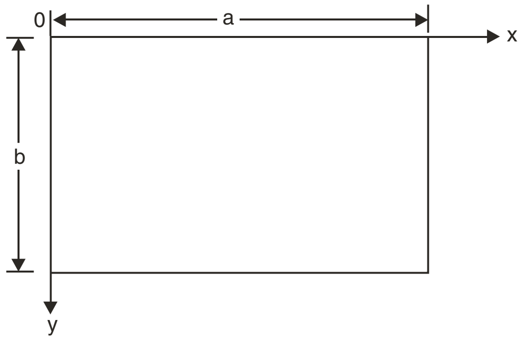


Fig. 4.7 Typical plate of size $a \times b$.

Since, $$g_m(y) = \sum_{n=1}^{\infty} a_{mn}\sin\frac{n\pi y}{b}$$

$$a_{mn} = \frac{2}{b}\int_0^b g_m(y)\sin\frac{n\pi y}{b}dy$$

$$= \frac{2}{b}\int_0^b \frac{2}{a}\int_0^a q(x, y)\sin\frac{m\pi x}{a}\cdot\sin\frac{n\pi y}{b}dx\,dy$$

i.e. $$a_{mn} = \frac{4}{ab}\int_0^a\int_0^b q(x,y)\sin\frac{m\pi x}{a}\cdot\sin\frac{n\pi y}{b}dx\,dy \qquad \text{...eqn. 4.9}$$

4.8 DOUBLE FOURIER FORM FOR UDL

For uniformly distributed load,

$$q(x, y) = q_0$$

$$\therefore \qquad a_{mn} = \frac{4}{ab}\int_0^a\int_0^b q_0\sin\frac{m\pi x}{a}\cdot\sin\frac{n\pi y}{b}\cdot dx\,dy$$

$$= \frac{4q_0}{ab}\left[-\frac{a}{m\pi}\cos\frac{m\pi x}{a}\right]_0^a \left[-\frac{b}{n\pi}\cos\frac{n\pi y}{b}\right]_0^b$$

$$= \frac{4q_0}{mn\pi^2}[-\cos m\pi + 1]\,[-\cos n\pi + 1]$$

∴ For even values of m or n, $a_{mn} = 0$.

For odd values of m and n, $a_{mn} = \frac{4q_0}{mn\pi^2} \times 2 \times 2 = \frac{16q_0}{mn\pi^2}$

∴ $$q(x, y) = \sum\sum \frac{16q_0}{mn\pi^2}\sin\frac{m\pi x}{a}\cdot\sin\frac{n\pi y}{b} \quad \text{...eqn. 4.10}$$

4.9 DOUBLE FOURIER FORM FOR UDL OVER SMALL AREA

Figure 4.8 shows a plate of size $a \times b$ subject to *udl* q_0 over a small area $u \times v$ with its centre of gravity at (ξ, η). Now, it is required to find Fourier equivalent for this load.

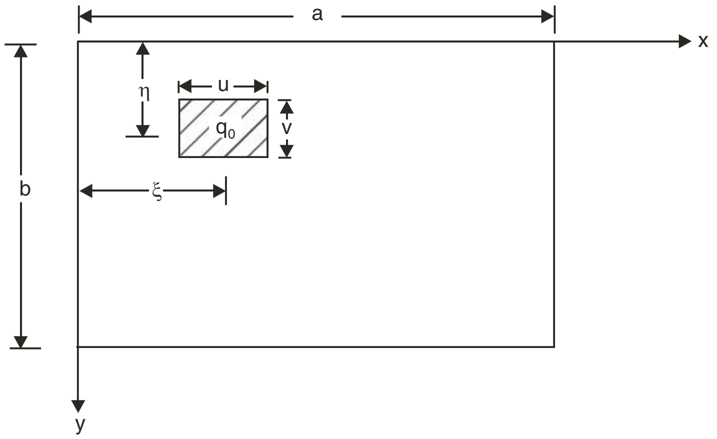


Fig. 4.8 UDL over a small area

For this load,

$q(x, y) = 0$ for $x = 0$ to $\xi - u/2$ and for $x = \xi + u/2$ to a

$= 0$ for $y = 0$ to $\eta - v/2$ and for $y = \eta + v/2$ to b

$= q_0$ for $x = \xi - u/2$ to $\xi + u/2$ and $y = \eta - v/2$ to $\eta + v/2$.

∴ $$a_{mn} = \frac{4}{ab}\int_0^a\int_0^b q(x, y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy$$

$$= \frac{4}{ab}\int_{\xi-u/2}^{\xi+u/2}\int_{\eta-v/2}^{\eta+v/2} q_0\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy$$

$$= \frac{4q_0}{ab}\left[-\frac{a}{m\pi}\cos\frac{m\pi x}{a}\right]_{\xi-u/2}^{\xi+u/2}\left[-\frac{b}{n\pi}\cos\frac{n\pi y}{b}\right]_{\eta-v/2}^{\eta+v/2}$$

$$= \frac{4q_0}{mn\pi^2}\left[\cos\frac{m\pi(\xi+u/2)}{a} - \cos\frac{m\pi(\xi-u/2)}{a}\right]$$

$$\left[\cos\frac{n\pi(\eta+v/2)}{b} - \cos\frac{n\pi(\eta-v/2)}{b}\right]$$

$$= \frac{4q_0}{mn\pi^2}\cdot 2\sin\frac{m\pi\xi}{a}\cdot\sin\frac{m\pi u}{2a}\cdot 2\sin\frac{n\pi n}{b}\cdot\sin\frac{n\pi v}{2b}$$

$$\therefore \quad q(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{16q_0}{mn\pi^2}\sin\frac{m\pi u}{2a}\cdot\sin\frac{m\pi\xi}{a}\cdot\sin\frac{n\pi v}{2b}\sin\frac{n\pi n}{b}\cdot\sin\frac{m\pi x}{a}\cdot\sin\frac{n\pi y}{b} \qquad \text{...eqn. 4.11}$$

4.10 DOUBLE FOURIER FORM FOR CONCENTRATED LOAD

Let P be the concentrated load acting at (ξ, η) on plate of size $a \times b$ as shown in Fig. 4.9.

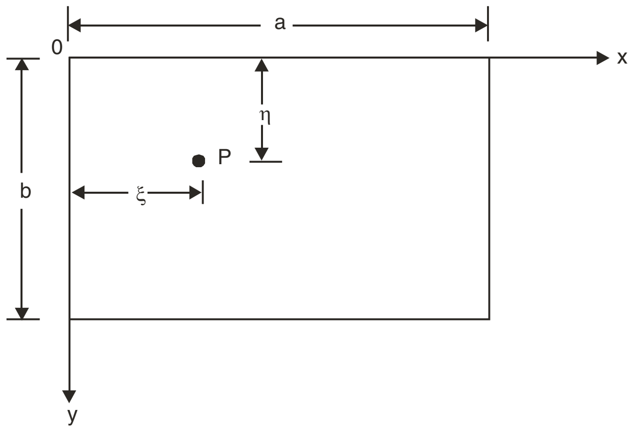


Fig. 4.9 Concentrated Load P at (ξ, η)

Double Fourier expression for this load can be derived from eqn. 4.11 by letting

$$P = q_0 uv$$

while letting u and v to tend to zero.

Since, u and v tend to zero,

$$\sin\frac{m\pi u}{2a} = \frac{m\pi u}{2a}$$

and

$$\sin\frac{n\pi v}{2b} = \frac{n\pi v}{2b}$$

$$\therefore \qquad q(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{16q_0}{mn\pi^2} \cdot \frac{m\pi u}{2a} \cdot \frac{n\pi v}{2b} \cdot \sin\frac{m\pi\xi}{a} \cdot \sin\frac{n\pi n}{b} \cdot \sin\frac{m\pi x}{a} \cdot \sin\frac{n\pi y}{b}.$$

i.e.
$$q(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{4P}{ab} \sin\frac{m\pi\xi}{a} \cdot \sin\frac{n\pi n}{b} \cdot \sin\frac{m\pi x}{a} \cdot \sin\frac{n\pi y}{b} \qquad \text{...eqn. 4.12}$$

4.11 DOUBLE FOURIER EQUIVALENT FOR HYDROSTATIC LOAD

Let loading vary linearly from zero to q_0 in x-direction and be constant in y-direction (Refer Fig. 4.10).

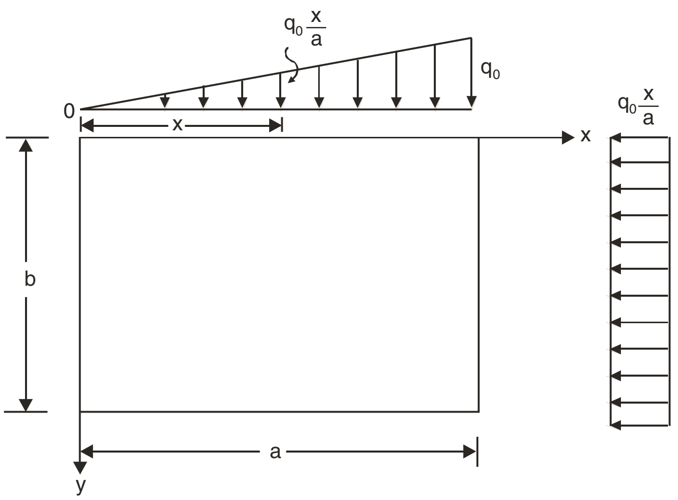


Fig. 4.10 Hydrostatic load on plate

$$q(x, y) = \sum\sum a_{mn} \sin\frac{m\pi x}{a} \cdot \sin\frac{n\pi y}{b}$$

where
$$a_{mn} = \frac{4}{ab}\int_0^a\int_0^b q(x, y) \sin\frac{m\pi x}{a} \cdot \sin\frac{n\pi y}{b}\, dx\, dy$$

$$= \frac{4}{ab}\int_0^a\int_0^b q_0 \frac{x}{a} \sin\frac{m\pi x}{a} \cdot \sin\frac{n\pi y}{b}\, dx\, dy$$

From eqn. 4.8, we know

$$\frac{2}{a}\int_0^a \frac{q_0 x}{a} \sin\frac{m\pi x}{a}\, dx = \frac{2q_0}{m\pi}(-1)^{m+1}.$$

$$\therefore \qquad a_{mn} = \frac{2}{b}\frac{2q_0}{m\pi}(-1)^{m+1}\int_0^b \sin\frac{n\pi y}{b}\, dy$$

$$= \frac{4q_0(-1)^{m+1}}{m\pi b}\left[-\frac{b}{n\pi}\cos\frac{n\pi y}{b}\right]_0^b$$

$$= \frac{4q_0(-1)^{m+1}}{mn\pi^2}[-\cos n\pi + 1]$$

= 0 for even values of m

$$= \frac{8q_0(-1)^{m+1}}{mn\pi^2} \text{ for odd values of } n.$$

$$\therefore \quad q(x, y) = \sum_{m=1}^{\infty}\sum_{n=1,3,\ldots}^{\infty} \frac{8q_0(-1)^{m+1}}{mn\pi^2}\sin\frac{m\pi x}{a}\cdot\sin\frac{n\pi y}{b}. \quad \text{...eqn. 4.13}$$

4.12 DOUBLE FOURIER EQUIVALENT FOR TRIANGULAR LOAD WITH PEAK VALUE AT MID-SPAN

Figure 4.11 shows the loading.

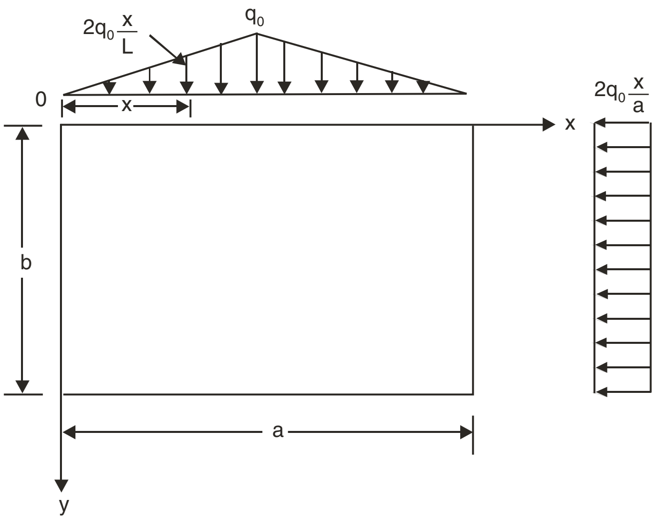


Fig. 4.11 Triangular load with peak value at mid-span

In this case

$$q(x, y) = \frac{2q_0 x}{a} \text{ for } x = 0 \text{ to } \frac{a}{2}$$

and the loading is symmetric

$$\therefore \quad \int_0^a q(x, y)\,dx = 2\int_0^{a/2} q(x, y)\,dx$$

$$\therefore \quad a_{mn} = \frac{4}{ab}\int_0^a\int_0^b q(x, y)\sin\frac{m\pi x}{a}\cdot\sin\frac{n\pi y}{b}\,dx\,dy$$

$$= \frac{4}{ab}\times 2\int_0^{a/2}\int_0^b \frac{2q_0 x}{a}\sin\frac{m\pi x}{a}\cdot\sin\frac{n\pi y}{b}\,dx\,dy$$

$$= \frac{16q_0}{a^2 b}\int_0^{a/2}\int_0^b x\sin\frac{m\pi x}{a}\cdot\sin\frac{n\pi y}{b}\,dx\,dy$$

From eqn. 4.8(*a*), we know

$$\int x\sin\frac{m\pi x}{a} = \frac{a^2}{m^2\pi^2}\sin\frac{m\pi}{2}$$

$$\therefore \quad a_{mn} = \frac{16q_0}{m^2\pi^2 b}\sin\frac{m\pi}{2}\int_0^b \sin\frac{n\pi y}{b}\,dy$$

$$= \frac{16q_0}{m^2\pi^2 b}\sin\frac{m\pi}{2}\left[-\frac{b}{n\pi}\cos\frac{n\pi y}{b}\right]_0^b$$

$$= \frac{16q_0}{m^2 n\pi^3}\sin\frac{m\pi}{2}\cdot[-\cos n\pi + 1]$$

$= 0$ for even values of n.

$$= \frac{32q_0}{m^2 n\pi^3}\sin\frac{m\pi x}{2} \text{ for odd values of } n$$

$$\therefore \quad q(x, y) = \sum_{m=1}^{\infty}\sum_{n=1,3,\ldots}^{\infty}\frac{32q_0}{m^2 n\pi^3}\sin\frac{m\pi}{2}\sin\frac{m\pi x}{a}\cdot\sin\frac{n\pi y}{b}. \qquad \text{...eqn. 4.14}$$

QUESTIONS

1. Derive single Fourier series equivalent for the following type of loads on beam of span L:
   (*a*) Uniformly distributed load over entire span.
   (*b*) Uniformly distributed load over a small length u with its centre of gravity at $x = \xi$.
   (*c*) Concentrated load P at $x = \xi$.
   (*d*) Hydrostatic load of maximum intensity q_0 at $x = L$ and zero intensity at $x = 0$.
   (*e*) Triangular load of intensity q_0 at $x = L/2$.
2. Derive double Fourier series expressions for the following loads:
   (*a*) Uniformly distributed load over entire plate.
   (*b*) Concentrated load P at (ξ, η).

Chapter

5

Navier's Solution for Rectangular Plates

A French engineer Navier presented a plate theory as back as in 1820. His theory holds good only for rectangular plates, simply supported along all the four edges. His paper was the beginning of looking for trigonometric series solutions for the analysis of plates. Navier's method is explained in this chapter for a simply supported rectangular plate subject to uniformly distributed load over entire plate. It is then extended for the plates subject to any other load.

5.1 NAVIER SOLUTION FOR RECTANGULAR PLATE SUBJECTED TO UDL

The uniformly distributed load q_0 may be expressed in its equivalent Fourier series load as,

$$q(x, y) = \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} \frac{16 q_0}{mn\pi^2} \sin\frac{m\pi x}{a} \cdot \sin\frac{n\pi y}{b}.$$

In this problem, the solution for w should satisfy the equation,

$$\nabla^4 w = \frac{q}{D} \; i.e. \; \frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D} \qquad \text{...eqn. 5.1}$$

and the boundary conditions

$w = 0$	at	$x = 0$	and	$x = a$
$w = 0$	at	$y = 0$	and	$y = b$
$M_x = 0$	at	$x = 0$	and	$x = a$, and
$M_y = 0$	at	$y = 0$	and	$y = b$.

Hence, let

$$w = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn} \sin\frac{m\pi x}{a} \cdot \sin\frac{n\pi y}{b} \qquad \text{...eqn. 5.2}$$

This form of deflection w satisfies all boundary conditions. But it has to satisfy the plate equation also. Now,

$$\frac{\partial^4 w}{\partial x^4} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn} \frac{m^4\pi^4}{a^4} \sin\frac{m\pi x}{a} \cdot \sin\frac{n\pi y}{b}$$

$$2\frac{\partial^4 w}{\partial x^2 \partial y^2} = 2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn} \frac{m^2\pi^2}{a^2} \cdot \frac{n^2\pi^2}{b^2} \sin\frac{m\pi x}{a} \cdot \sin\frac{n\pi y}{b}$$

and
$$\frac{\partial^4 w}{\partial y^4} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn}\frac{n^4\pi^4}{b^4}\sin\frac{m\pi x}{a}\cdot\sin\frac{n\pi y}{b}$$

Hence, if equivalent Fourier loading is considered, the plate equation reduces to

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn}\left[\frac{m^4\pi^4}{a^4}+\frac{2m^2n^2\pi^4}{a^2b^2}+\frac{n^4\pi^4}{b^4}\right]\sin\frac{m\pi x}{a}\cdot\sin\frac{n\pi y}{b}$$

$$=\frac{1}{D}\sum_{m=1,\,3,\,\ldots}^{\infty}\ \sum_{n=1,3,\ldots}^{\infty}\frac{16q_0}{mn\pi^2}\sin\frac{m\pi x}{a}\cdot\sin\frac{n\pi y}{b}$$

Comparing term by term, we get

$C_{mn} = 0$ for even values of m or n.

For odd values of m and n, we get

$$C_{mn}\left[\frac{m^4\pi^4}{a^4}+\frac{2m^2n^2\pi^4}{a^2b^2}+\frac{n^4\pi^4}{b^4}\right]=\frac{16q_0}{mn\pi^2 D}$$

i.e.
$$C_{mn}\frac{\pi^4}{a^4}\left[m^4+2m^2n^2\frac{a^2}{b^2}+\frac{n^4a^4}{b^4}\right]=\frac{16q_0}{mn\pi^2 D}$$

$\therefore$
$$C_{mn}=\frac{16q_0a^4}{mn\pi^6 D}\frac{1}{\left(m^2+\frac{a^2}{b^2}n^2\right)^2}$$

$\therefore$
$$w=\frac{16q_0a^4}{\pi^6 D}\sum_{m=1,3,\ldots}^{\infty}\ \sum_{n=1,3,\ldots}^{\infty}\frac{1}{mn}\frac{1}{\left(m^2+\frac{a^2}{b^2}n^2\right)^2}\sin\frac{m\pi x}{a}\cdot\sin\frac{n\pi y}{b} \qquad \text{...eqn. 5.3}$$

$\therefore$
$$M_x=-D\left(\frac{\partial^2 w}{\partial x^2}+\mu\frac{\partial^2 w}{\partial y^2}\right)$$

$$=-D\frac{16q_0a^4}{\pi^6 D}\sum_{m=1,3,\ldots}^{\infty}\ \sum_{n=1,3,\ldots}^{\infty}\frac{1}{mn}\frac{-\left\{\frac{m^2\pi^2}{a^2}+\mu\frac{n^2\pi^2}{b^2}\right\}}{\left(m^2+\frac{a^2}{b^2}n^2\right)^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

i.e.
$$M_x=\frac{16q_0a^2}{\pi^4}\sum_{m=1,3,\ldots}^{\infty}\ \sum_{n=1,3,\ldots}^{\infty}\frac{1}{mn}\frac{m^2+\mu n^2\frac{a^2}{b^2}}{\left(m^2+\frac{a^2}{b^2}n^2\right)^2}\sin\frac{m\pi x}{a}\cdot\sin\frac{n\pi y}{b} \qquad \text{...eqn. 5.4}$$

Similarly,
$$M_y=\frac{16q_0a^2}{\pi^4}\sum_{m=1,3,\ldots}^{\infty}\ \sum_{n=1,3,\ldots}^{\infty}\frac{1}{mn}\frac{\mu m^2+n^2\frac{a^2}{b^2}}{\left(m^2+n^2\frac{a^2}{b^2}\right)^2}\sin\frac{n\pi x}{a}\cdot\sin\frac{n\pi y}{b} \qquad \text{...eqn. 5.5}$$

Thus, expressions for all other stress resultants (M_{xy}, Q_x, Q_y, V_x, etc.) may be assembled easily.
For square plate $a = b$,

$$w = \frac{16 q_0 a^4}{\pi^6 D} \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} \frac{1}{mn} \frac{1}{\left(m^2+n^2\right)^2} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{a} \quad \text{...eqn. 5.6}$$

$$M_x = \frac{16 q_0 a^2}{\pi^4} \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} \frac{1}{mn} \frac{m^2+\mu n^2}{(m^2+n^2)^2} \sin\frac{m\pi x}{a} \cdot \sin\frac{n\pi y}{a} \quad \text{...eqn. 5.7}$$

$$M_y = \frac{16 q_0 a^2}{\pi^4} \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} \frac{1}{mn} \frac{\mu m^2+n^2}{(m^2+n^2)^2} \sin\frac{m\pi x}{a} \cdot \sin\frac{n\pi y}{a} \quad \text{...eqn. 4.8}$$

$$w_{\text{centre}} = \frac{16 q_0 a^4}{\pi^6 D} \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} \frac{1}{mn} \frac{1}{\left(m^2+n^2\right)^2} \sin\frac{m\pi}{2} \sin\frac{n\pi}{2}$$

$$= \frac{16 q_0 a^4}{\pi^6 D} \left[\frac{1}{1\times1(1+1)^2} - \frac{1}{3\times1(3^2+1)^2} - \frac{1}{1\times3(1+3^2)^2} + \frac{1}{3\times3} + \frac{1}{(3^2+3^2)^2} - \ldots \right]$$

$$= \frac{16 q_0 a^4}{\pi^6 D} \left[\frac{1}{4} - \frac{1}{3\times100} - \frac{1}{3\times100} + \frac{1}{9\times324} - \ldots \right]$$

$$w_{\text{centre, exact}} = 0.00406 \frac{q_0 a^4}{D}$$

The first term gives $w_{\text{centre}} = \frac{16}{\pi^6} \times \frac{1}{4} \frac{q_0 a^4}{D} = 0.0041606 \frac{q_0 a^4}{D}$

Thus, first term gives $\frac{0.0041606}{0.00406} \times 100 =$ 102.48 percent of total deflection

In other words, first term gives only 2.48% erroneous result and it is on safer side.
Consider **moment at centre of square plate taking μ = 0.**

$$M_x = M_y = \frac{16 q_0 a^2}{\pi^4} \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} \frac{1}{mn} \frac{m^2}{\left(m^2+n^2\right)^2} \sin\frac{m\pi}{2} \cdot \sin\frac{n\pi}{2}.$$

$$\text{First term} = \frac{1}{1\times1} \frac{1}{(1+1)^2} \cdot \frac{16 q_0 a^2}{\pi^4} = \frac{1}{4} \cdot \frac{16 q_0 a^2}{\pi^4}$$

$$\text{Second term} = -\frac{1}{3\times1} \frac{1\times9}{\left(3^2+1\right)^2} \cdot \frac{16 q_0 a^2}{\pi^4} = -\frac{3}{100} \frac{16 q_0 a^2}{\pi^4}$$

$$\text{Third term} = -\frac{1}{1\times3} \frac{1}{\left(1\times3^2\right)^2} \frac{16 q_0 a^2}{\pi^4} = -\frac{1}{300} \frac{16 q_0 a^2}{\pi^4}$$

$$\text{Fourth term} = \frac{5^2}{5\times 26^2}\frac{16q_0a^2}{\pi^4} = \frac{5}{3380}\frac{16q_0a^2}{\pi^4}$$

$$\text{Fifth term} = \frac{1^2}{5\times 26^2}\frac{16q_0a^2}{\pi^4} = \frac{1}{3380}\frac{16q_0a^2}{\pi^4}$$

It may be observed that convergence is not that fast as deflection w. Hence, to get results within 1.1 percent errors only, we have to take at least 3 terms.

5.2 NAVIER SOLUTION FOR ANY LOADING

Navier solution for simply supported rectangular plate subject to any loading can be easily found, if the load is expressed in the double Fourier form as

$$q(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \sin\frac{m\pi x}{a}\cdot\sin\frac{n\pi y}{b}$$

If we select

$$w = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn} \sin\frac{m\pi x}{a}\cdot\sin\frac{n\pi y}{b}$$

all boundary conditions are satisfied. Then from plate equation we get

$$C_{mn} = \frac{a_{mn}a^4}{D\pi^4\left(m^2 + n^2\dfrac{a^2}{b^2}\right)^2}$$

$$\therefore \quad w = \sum_{m=1,3,\ldots}^{\infty}\ \sum_{n=1,3,\ldots,}^{\infty} \frac{a_{mn}a^4}{\pi^4 D\left(m^2 + n^2\dfrac{a^2}{b^2}\right)^2}\sin\frac{m\pi x}{a}\cdot\sin\frac{n\pi y}{b}$$

Thus, if load is expressed in double Fourier form solution is readily available. Depending upon the loading, rate of convergence may slightly vary.

QUESTIONS

1. Derive the expression for deflection in a slightly bent simply supported rectangular plate subject to uniformly distributed load over entire plate. Use Navier's method. Determine the expressions for moments M_x, M_y, M_{xy} and shears Q_x, Q_y and V_x and V_y.
2. The Fourier equivalent load on a plate of size $a \times y$ is

$$q(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \sin\frac{m\pi x}{a}\cdot\sin\frac{n\pi y}{b}.$$

Derive the expressions for the following, if the plate is simply supported along all edges.
w, M_x, M_y, M_{xy}, Q_x, Q_y, V_x and V_y.

Chapter

6

Levy's Solution for Rectangular Plate Analysis

Navier's method is slow because it consists of summation of terms in two directions. Levy suggested a method in which summation of the Fourier terms in only one direction is required and the boundary conditions at the other two opposite edges are satisfied by closed form function. As series solution is in only one direction, it is converging faster than Navier's solution. This method is directly applicable for a plate with at least two opposite edges simply supported and the boundary conditions on other two opposite edges are any. The analysis is little bit lengthy but with only a few terms sufficiently accurate results can be obtained.

In this chapter, first application of Levy's method to simply supported rectangular plate is explained. Then its extension to rectangular plate with various end conditions is explained.

6.1 ANALYSIS OF RECTANGULAR PLATE SUBJECTED TO UDL BY LEVY'S METHOD

The Fourier equivalent load in x-directions corresponding to uniformly distributed load is

$$q = \sum_{m=1,3,\ldots}^{\infty} \frac{4q_0}{m\pi} \sin \frac{m\pi x}{a}$$

Let the deflection function for a plate of size $a \times b$ be,

$$w = \sum_{m=1}^{\infty} Y_m \sin \frac{m\pi x}{a}$$

where Y_m is a function of y only.

It satisfies the boundary conditions at $x = 0$ and at $x = a$ (*i.e.* $w = 0$; $M_x = 0$ at $x = 0$ and $x = a$).

Now we have to select Y_m such that it satisfies the boundary conditions at $y = 0$ and $y = b$ and also the plate equation $\nabla_w^4 = \frac{q}{D}$

Now, $$\frac{\partial^4 w}{\partial x^4} = \sum Y_m \frac{m^4 \pi^4}{a^4} \sin \frac{m\pi x}{a}$$

$$2\frac{\partial^4 w}{\partial x^2 \partial y^2} = \sum -2Y_m'' \frac{m^2\pi^2}{a^2}\sin\frac{m\pi x}{a}$$

$$\frac{\partial^4 w}{\partial y^4} = \sum Y_m'''' \sin\frac{m\pi x}{a}$$

where $$Y_m'' = \frac{\partial^2 Y_m}{\partial y^2} \text{ and } Y_m'''' = \frac{\partial^4 Y_m}{\partial y^4}.$$

∴ The plate equation for this case is

$$\sum_{m=1}^{\infty}\left(\frac{m^4\pi^4}{a^4}Y_m - 2\frac{m^2\pi^2}{a^2}Y_m'' + Y_m''''\right)\sin\frac{m\pi x}{a} = \frac{1}{D}\sum_{m=1,3,\ldots}^{\infty}\frac{4q_0}{m\pi}\sin\frac{m\pi x}{a}$$

Comparing term by term, even terms on left hand side vanish and for odd terms, we get;

$$\frac{m^4\pi^4}{a^4}Y_m - 2\frac{m^2\pi^2}{a^2}Y_m'' + Y_m''' = \frac{1}{D}\frac{4q_0}{m\pi} \qquad \text{...eqn. 6.1}$$

When the above equation is solved there will be four constants of integration and we have got four boundary conditions *i.e.* two at $y = 0$ and two at $y = b$. Hence, the problem can be solved.

Solution of the equation:

It consists of particular solution and complementary solution.

Particular solution:

The equation 6.1 can be written as

$$\left(\frac{m^2\pi^2}{a^2} - \partial^2\right)^2 Y_m = \frac{4q_0}{m\pi D}$$

∴ $$Y_{m,\text{particular}} = \frac{4q_0}{m\pi D}\left(\frac{m^2\pi^2}{a^2} - \partial^2\right)^{-2}$$

$$= \frac{4q_0}{m\pi D}\left(\frac{m^2\pi^2}{a^2}\right)^{-2}\left(1 - \frac{a^2}{m^2\pi^2}\partial^2\right)^{-2}$$

$$= \frac{4q_0 a^4}{m^5\pi^5 D}\left(1 + 2\frac{a^2}{m^2\pi^2}\partial^2 + \ldots\right)$$

$$= \frac{4q_0 a^4}{m^5\pi^5 D}\left[1 + \partial^2\left(2\frac{a^2}{m^2\pi^2}\right)\right]$$

$$= \frac{4q_0 a^4}{m^5\pi^5 D}$$

Complementary Integral:

$$\left(\frac{m^2\pi^2}{a^2} - \partial^2\right)^2 Y_m = 0\,.$$

The repeated roots are

$$\frac{m^2\pi^2}{a^2} - \partial^2 = 0$$

$$\partial = \pm\frac{m\pi}{a}\,.$$

Hence, the complementary integral is

$$Y_m = H_1 e^{\frac{m\pi}{a}y} + H_2 e^{\frac{-m\pi}{a}y} + H_3 y e^{\frac{m\pi y}{a}} + H_4 y e^{\frac{-m\pi y}{a}}$$

Since, H_1, H_2, H_3 and H_4 are arbitrary constants to be determined from boundary conditions, let us select some other arbitrary constants A_m, B_m, C_m and D_m such that,

$$H_1 = \frac{A_m + C_m}{2},\; H_2 = \frac{A_m - C_m}{2}$$

$$H_3 = \frac{B_m + D_m}{2} m\pi \text{ and } H_4 = \frac{-B_m + D_m}{2}\cdot m\pi$$

Hence,

$$Y_m = \frac{A_m + C_m}{2} e^{\frac{m\pi y}{a}} + \frac{A_m - C_m}{2} e^{\frac{-m\pi y}{a}}$$

$$+ \frac{B_m + D_m}{2}\frac{m\pi y}{a} e^{\frac{m\pi y}{a}} + \frac{-B_m + D_m}{2}\frac{m\pi y}{a}\cdot e^{\frac{-m\pi y}{a}}$$

$$= A_m\left(\frac{e^{\frac{m\pi y}{a}} + e^{\frac{-m\pi y}{a}}}{2}\right) + C_m \frac{e^{\frac{m\pi y}{a}} - e^{\frac{-m\pi y}{a}}}{2}$$

$$+ B_m \frac{m\pi y}{a}\cdot\frac{e^{\frac{m\pi y}{a}} - e^{\frac{-m\pi y}{a}}}{2} + D_m \frac{m\pi y}{a}\left(\frac{e^{\frac{m\pi y}{a}} + e^{\frac{-m\pi y}{a}}}{2}\right)$$

Thus,

$$y_m = A_m \cosh\frac{m\pi y}{a} + B_m \frac{m\pi y}{a}\sinh\frac{m\pi y}{a}$$

$$+ C_m \sinh\frac{m\pi y}{a} + D_m\frac{m\pi y}{a}\cosh\frac{m\pi y}{a} \qquad \text{...eqn. 6.2}$$

It is convenient to select the origin of the coordinate system *x, y* at the point of symmetry as shown in Fig. 6.1.

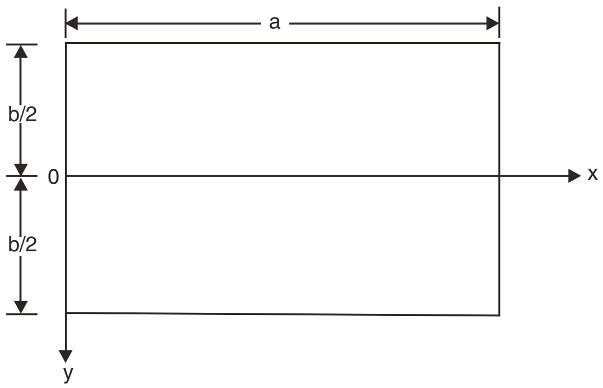


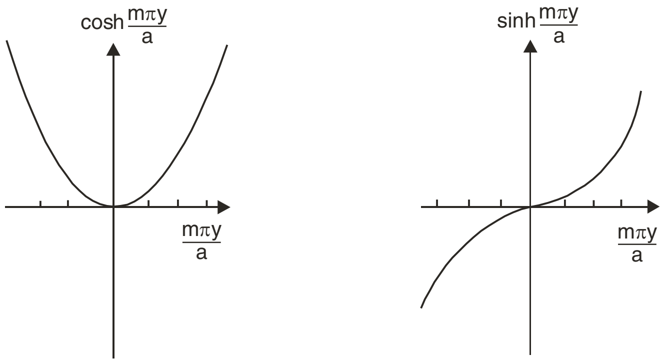
Fig. 6.1

It may be observed that,

$$\cosh\frac{m\pi y}{a} = \frac{e^{\frac{m\pi y}{a}} + e^{\frac{-m\pi y}{a}}}{2}$$

and $$\cosh\{-m\pi(-y)\} = \frac{e^{\frac{-m\pi y}{a}} + e^{\frac{m\pi y}{a}}}{2} = \cosh\frac{m\pi y}{a}.$$

Thus, $\cosh\frac{m\pi y}{a}$ is a symmetric function.



(*a*) Variation of cosh $\frac{m\pi y}{a}$ (*b*) Variation of sinh $\frac{m\pi y}{a}$

Fig. 6.2

Similarly,

$$\sinh\frac{m\pi y}{a} = \frac{e^{\frac{m\pi y}{a}} - e^{-\frac{m\pi y}{a}}}{2}$$

and $$\sin\frac{m\pi(-y)}{a}=\frac{e^{\frac{-m\pi y}{a}}-e^{\frac{m\pi y}{a}}}{2}=-\sinh\frac{m\pi y}{a}.$$

Hence, $\sinh\left(\frac{m\pi y}{a}\right)$ is antisymmetric function. Figure 6.2 shows the variation of $\cosh\frac{m\pi y}{a}$ and $\sinh\frac{m\pi y}{a}$.

Similarly, it may be observed that $\frac{m\pi y}{a}\sinh\frac{m\pi y}{a}$ is symmetric term and $\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}$ is anti-symmetric term.

Hence, when symmetric load acts, there cannot be antisymmetric terms in deflection.

i.e. For symmetric loading, homogeneous solution is

$$Y_m = A_m\cosh\frac{m\pi y}{a}+B_m\frac{m\pi y}{a}\sinh\frac{m\pi y}{a} \qquad \text{...eqn. 6.3}$$

Similarly *for antisymmetric loading,* homogeneous solution is

$$Y_m = C_m\frac{m\pi y}{a}\sinh\frac{m\pi y}{a}+D_m\frac{m\pi y}{a}\cosh\frac{m\pi y}{a} \qquad \text{...eqn. 6.4}$$

The arbitrary constants are to be determined using boundary conditions.

Thus we have

$$Y = Y_{\text{particular}} + Y_{\text{homogeneous}}$$

For *udl,* since load is symmetric

$$Y_m=\frac{4q_0a^4}{m^5\pi^5D}+A_m\cosh\frac{m\pi y}{a}+B_m\frac{m\pi y}{a}\sinh\frac{m\pi y}{a}.$$

Redefining the arbitrary constants, let us take

$$Y_m=\frac{q_0a^4}{D}\left[\frac{4}{m^5\pi^5}+A_m\cosh\frac{m\pi y}{a}+B_m\frac{m\pi y}{a}\sinh\frac{m\pi y}{a}\right]$$

and hence $$w=\sum_{m=1,3,\ldots}^{\infty} F_{mt}-y_m\sin\frac{m\pi x}{a}$$

The boundary conditions are

at $y = \pm b/2$, $w = 0$ ...(1)

and $M_y = 0$

i.e. $$-D\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)_{y\pm\frac{b}{2}}=0$$

but $\frac{\partial^2 w}{\partial x^2}=0,$ since it is supported all along $y = \pm b/2$.

Hence, the second boundary condition is

$$-D\frac{\partial^2 w}{\partial y^2}\bigg|_{y=\pm\frac{b}{2}} = 0 \qquad \text{...(2)}$$

From first boundary condition for all odd values of *m*,

$$\frac{4}{m^5\pi^5} + A_m \cosh\frac{m\pi b}{2a} + B_m \frac{m\pi b}{2a}\sinh\frac{m\pi b}{2a} = 0$$

Substituting α_m for $\frac{m\pi b}{2a}$, the first boundary condition is

$$\frac{4}{m^5\pi^5} + A_m \cosh\alpha_m + B_m\alpha_m \sinh\alpha_m = 0 \qquad \text{...(3)}$$

Now, $$Y_m' = \frac{dY_m}{dy} = \left[A_m\frac{m\pi}{a}\sinh\frac{m\pi y}{a} + B_m\frac{m\pi}{a}\sinh\frac{m\pi y}{a} + B_m\frac{m\pi y}{a}\frac{m\pi}{a}\cosh\frac{m\pi y}{a}\right]\frac{q_0a^4}{D}$$

$$= \frac{q_0a^4}{D}\frac{m\pi}{a}\left[(A_m + B_m)\sinh\frac{m\pi y}{a} + B_m\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\right]$$

$$Y_m'' = \frac{d^2Y_m}{dy^2} = \frac{q_0a^4}{D}\cdot\frac{m\pi}{a}\left[(A_m + B_m)\frac{m\pi}{a}\cosh\frac{m\pi y}{a} + B_m\left(\frac{m\pi}{a}\cdot\cosh\frac{m\pi y}{a} + \frac{m\pi y}{a}\frac{m\pi}{a}\sinh\frac{m\pi y}{a}\right)\right]$$

$$= \frac{q_0a^4}{D}\frac{m^2\pi^2}{a^2}\left[(A_m + 2B_m)\cosh\frac{m\pi y}{a} + B_m\frac{m\pi y}{a}\sinh\frac{m\pi y}{a}\right]$$

∴ From boundary condition (2), we get

$$(A_m + 2B_m)\cosh\frac{m\pi b}{a} + B_m\frac{m\pi b}{a}\sinh\frac{m\pi b}{a} = 0$$

i.e. $$(A_m + 2B_m)\cosh\alpha_m + B_m\alpha_m \sinh\alpha_m = 0 \qquad \text{...(4)}$$

Subtracting eqn. (3) from eqn. (4), we get

$$2B_m\cosh\alpha_m - \frac{4}{m^5\pi^5} = 0$$

or $$B_m = \frac{2}{m^5\pi^5\cosh\alpha_m}$$

Substituting this value of B_m in eqn. (3), we get

$$\frac{4}{m^5\pi^5} + A_m\cosh\alpha_m + \frac{2}{m^5\pi^5\cosh\alpha_m}\alpha_m\sinh\alpha_m = 0$$

∴ $$A_m\cosh\alpha_m = -\frac{2}{m^5\pi^5}\left[2 + \alpha_m\tanh\alpha_m\right]$$

$$\therefore \qquad A_m = -\frac{2}{m^5\pi^5 \cosh\alpha_m}\left[2 + \alpha_m \tanh\alpha_m\right]$$

$$\therefore \; w = \sum_{m=1,3,\ldots}^{\infty} \frac{q_0 a^4}{D}\left[\frac{4}{m^5\pi^5} - \frac{2\left(2+\alpha_m \tanh\alpha_m\right)}{m^5\pi^5 \cosh\alpha_m}\cosh\frac{m\pi y}{a} + \frac{2}{m^5\pi^5 \cosh\alpha_m}\frac{m\pi y}{a}\sinh\frac{m\pi y}{a}\right]\sin\frac{m\pi x}{a}$$

Thus, $$w = \sum_{m=1,3,\ldots}^{\infty} \frac{4q_0 a^4}{m^5\pi^5 D}\left[1 - \frac{2+\alpha_m \tanh\alpha_m}{2\cosh\alpha_m}\cosh\frac{m\pi y}{a} + \frac{1}{2\cosh\alpha_m}\frac{m\pi y}{a}\sinh\frac{m\pi y}{a}\right]\sin\frac{m\pi x}{a}$$

...eqn. 6.5

The above expression for deflected surface is fast converging and hence only a few terms give sufficiently accurate results. The main reason for fast convergence in one of the series is already summed up.

6.2 PHYSICAL MEANING OF PARTICULAR AND COMPLEMENTARY (HOMOGENEOUS) INTEGRALS

Consider a beam of span 'a' subject to uniformly distributed load q_0 over its entire span. Hence its deflection is given by

$$D\frac{\partial^4 w}{\partial x^4} = q_0$$

Expressing the load in Fourier form,

$$D\frac{\partial^4 w}{\partial x^4} = \sum_{m=1,3,\ldots}^{\infty} \frac{4q_0}{m\pi}\sin\frac{m\pi x}{a}$$

$$\therefore \qquad w = \sum_{m=1,3,\ldots}^{\infty} \frac{4q_0}{m^5\pi^5 D}\cdot\sin\frac{m\pi x}{a}$$

This is same as the first term in deflection expression for plate which is due to particular integral. Hence, physical meaning of particular integral is, it gives deflection of the plate which has free boundary conditions at edges $y = \pm b/2$. In other words, particular solution is due to cylindrical bending of the plate. Hence, the complimentary integral gives effect of boundary conditions. Thus, for plate

$$w = w_1 + w_2$$

where w_1 – deflection of strip unaffected by boundary conditions at $y = \pm b/2$
and w_2 – effect of boundary conditions at $y = \pm b/2$. (Complimentary integral contribution).

6.3 CONVERGENCE STUDY

Maximum deflection occurs at middle of plate *i.e.* at $x = \frac{a}{2}$ and $y = 0$.

$$w_{\text{centre}} = w_{\text{centre of strip}} + w_{\text{centre due to complimentary integral}}$$

We know,

$$w_{\text{centre of strip}} = \frac{5}{384}\frac{q_0 a^4}{D}$$

$$\therefore \qquad w_{\text{centre}} = \frac{5}{384}\frac{q_0 a^4}{D} - \sum_{m=1,3,\ldots}^{\infty} \frac{2+\alpha_m \tanh \alpha_m}{2\cosh \alpha_m} \sin \frac{m\pi}{2}$$

For a square plate,

$$a = b \;\therefore\; \alpha_m = \frac{m\pi b}{2a} = \frac{m\pi}{2}$$

$$\therefore \qquad w = \frac{5}{384}\frac{q_0 a^4}{D} - \frac{4q_0 a^4}{\pi^5 D}[0.68562 - 0.00025 + \ldots]$$

$$= 0.00406 \frac{q_0 a^4}{D}$$

It may be observed that second term in the series is less than 0.04 percentage of first term in the series. Hence, the first term itself gives more than 99.96 percentage accuracy.

On the same lines moment expressions also may be studied. Study of bending moment values for different $\frac{b}{a}$ ratio shows:

1. For $\frac{b}{a} > 3,$ the calculation for a plate may be replaced by those for strip.
2. If $\frac{b}{a} = 2,$ the bending moment at middle of the plate is over estimated by 18.6 percent by strip theory. Hence, many codes recommend design of slabs treating them as one way slabs if $\frac{b}{a} > 2.$ Note that by this approximation error is on safer side.

6.4 COMPARISON BETWEEN NAVIER'S SOLUTION AND LEVY'S SOLUTION

From study of the above two methods the following points may be observed.

1. Navier's solutions are simple while Levy's solutions are lengthy.
2. To get satisfactory results more terms in series should be considered when Navier's method is used while Levy's method gives satisfactory results with few terms only.
3. Navier's solution is applicable for plate with all four edges simply supported whereas Levy's method holds good for a plate with two opposite edges simply supported and other two having any conditions.
4. Levy's method can be extended for the analysis of plates with any edge conditions whereas it is not possible to use Navier's solutions for plate with boundary conditions other than all four edges simply supported.

QUESTION

1. Derive the expressions for deflection for a simply supported rectangular plate of size $a \times b$. Use Levy's method.

Chapter

7

Rectangular Plates with Various Edge Conditions

In general, we come across plates with various edge conditions. Some edges may be free, some fixed and some other clamped. It is possible to extend plate theory to all possible combinations of edge conditions. In this chapter, method of analysing plates with various combination of edge conditions is discussed. First, a case of a plate with all edges simply supported and subject to moments $f_1(x)$ and $f_2(x)$ along edges $y = 0$ and $y(b)$ is presented. Then using this solution with other solutions is discussed for getting solution for plates with different edge conditions.

7.1 SIMPLY SUPPORTED PLATE SUBJECT TO MOMENTS ALONG $y = 0$ AND $y = b$

Figure 7.1 shows a rectangular plate of size $a \times b$ simply supported along all four edges and subjected to moments $f_1(x)$ along $y = -b/2$ edge and $f_2(x)$ along the edge $y = b/2$.

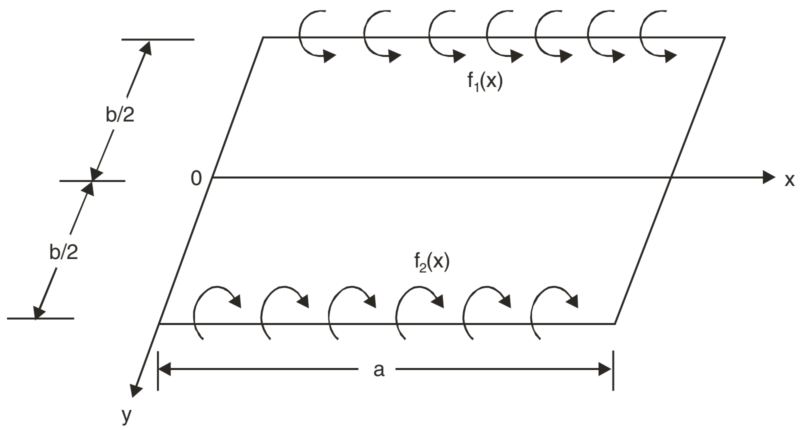


Fig. 7.1 Plate subjected to edge moments

For this case the deflection function w should satisfy the plate equation

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D} = 0 \qquad \text{...eqn. 7.1}$$

and also the following boundary conditions.

$$w = 0 \text{ and } \frac{\partial^2 w}{\partial x^2} = 0 \text{ for the edges } x = 0 \text{ and } x = a$$

$$w = 0 \text{ at } y = \pm b/2 \qquad \text{...eqn. 7.2}$$

$$-D\left(\frac{\partial^2 w}{\partial y^2}\right)_{y=-\frac{b}{2}} = f_1(x) \qquad \text{...eqn. 7.3}$$

and $$-D\left(\frac{\partial^2 w}{\partial y^2}\right)_{y=\frac{b}{2}} = f_2(x) \qquad \text{...eqn. 7.4}$$

In this case as $q(x, y) = 0$, particular integral is zero. Hence, Levy's solution is

$$w = \Sigma Y_m \sin\frac{m\pi x}{a}$$

where $$Y_m = A_m \cosh\frac{m\pi y}{a} + B_m \frac{m\pi y}{a}\sinh\frac{m\pi y}{a} + C_m \sinh\frac{m\pi y}{a} + D_m \frac{m\pi y}{a}\cosh\frac{m\pi y}{a}$$

If $f_1(x)$ and $f_2(x)$ are not the same, it is possible to split the case into a symmetric case of loading $\frac{f_1(x)+f_2(x)}{2}$ and an anti-symmetric case of loading $\frac{f_1(x)-f_2(x)}{2}$ as shown in Figure 7.2. For example, if $f_1(x)$ is a uniform moment of 80 kN-m and $f_2(x)$ = 60 kN-m it may be split into a symmetric uniform moment of $\frac{80+60}{2} = 70$ kN-m and an anti-symmetric uniform moment of $\frac{80-60}{2} = 10$ kN-m. Hence it is better to study the analysis for a symmetric moment $\Sigma E_m \sin\frac{m\pi x}{a}$ and for an anti-symmetric moment $\Sigma E'_m \sin\frac{m\pi x}{a}$.

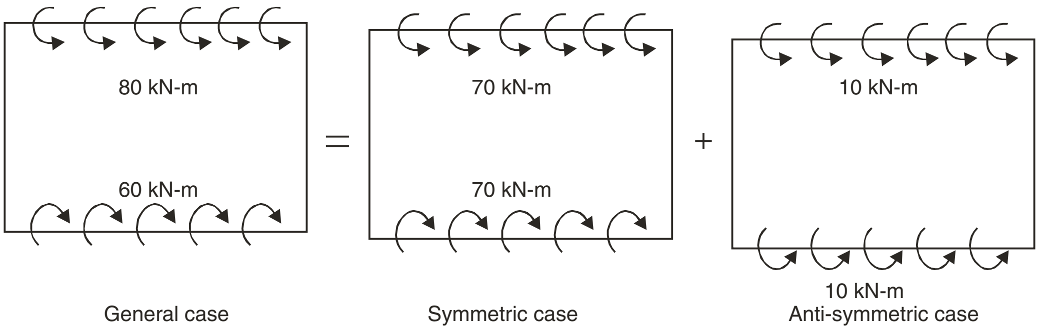


Fig. 7.2

(*a*) *Symmetric Moment Case*

Figure 7.3 shows this case.

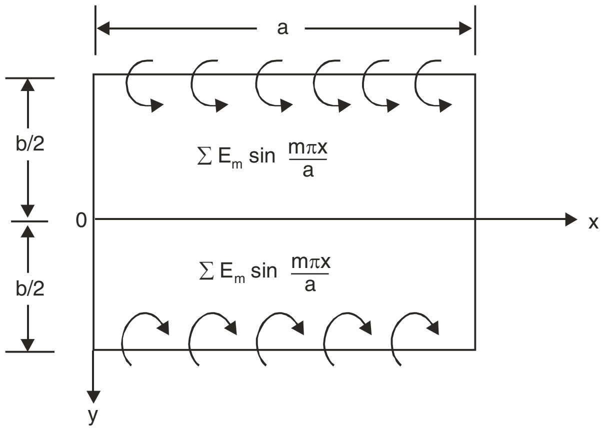


Fig. 7.3 Plate with symmetric edge moments

Since, the loading is symmetric, the anti-symmetric terms in deflection expression must vanish *i.e.* $C_m = D_m = 0$.

$$\therefore \qquad w = \sum \left(A_m \cosh \frac{m\pi y}{a} + B_m \frac{m\pi y}{a} \sinh \frac{m\pi y}{a} \right) \sin \frac{m\pi x}{a}$$

The boundary condition $w = 0$ at $y = \pm b/2$ gives

$$0 = A_m \cosh \frac{m\pi b}{2a} + B_m \frac{m\pi b}{2a} \sinh \frac{m\pi b}{2a}$$

Substituting α_m for $\dfrac{m\pi b}{2a}$, we can write

$$0 = A_m \cosh \alpha_m + B_m \alpha_m \sinh \alpha_m \qquad \text{...(1)}$$

Now $$\frac{\partial w}{\partial y} = \sum \frac{m\pi}{a} \left(A_m \sinh \frac{m\pi y}{a} + B_m \sinh \frac{m\pi y}{a} + B_m \frac{m\pi y}{a} \cosh \frac{m\pi y}{a} \right) \sin \frac{m\pi x}{a}$$

$$\frac{\partial^2 w}{\partial y^2} = \sum \frac{m^2\pi^2}{a^2} \left(A_m \cosh \frac{m\pi y}{a} + B_m \cosh \frac{m\pi y}{a} + B_m \cosh \frac{m\pi y}{a} + B_m \frac{m\pi y}{a} \cdot \sinh \frac{m\pi y}{a} \right) \sin \frac{m\pi x}{a}$$

$$= \sum \frac{m^2\pi^2}{a^2} \left(A_m \cosh \frac{m\pi y}{a} + 2B_m \cosh \frac{m\pi y}{a} + B_m \frac{m\pi y}{a} \sinh \frac{m\pi y}{a} \right) \sin \frac{m\pi x}{a}$$

Hence, boundary condition

$$M_y = \sum E_m \sin \frac{m\pi x}{a}$$

at $y = \pm b/2$ means

$$-D\left.\frac{\partial^2 w}{\partial y^2}\right|_{y=\pm\frac{b}{2}} = \sum E_m \sin\frac{m\pi x}{a}$$

Comparing term by term,

$$-D\frac{m^2\pi^2}{a^2}\left(A_m\cosh\alpha_m + 2B_m\cosh\alpha_m + B_m\alpha_m\sinh\alpha_m\right) = E_m$$

or $$A_m\cosh\alpha_m + 2B_m\cosh\alpha_m + B_m\alpha_m\sinh\alpha_m = -\frac{a^2}{m^2\pi^2 D}E_m \qquad ...(2)$$

Subtracting equation (1) from equation (2), we get

$$2B_m\cosh\alpha_m = -\frac{a^2}{m^2\pi^2 D}E_m$$

$\therefore$ $$B_m = -\frac{a^2 E_m}{2m^2\pi^2 D\cosh\alpha_m}$$

From eqn. (1)

$$A_m = -B_m\alpha_m\tanh\alpha_m$$

Substituting the value of B_m, we get

$$A_m = \frac{a^2 E_m}{2m^2\pi^2 D\cosh\alpha_m}\alpha_m\tanh\alpha_m$$

$\therefore$ $$w = \frac{a^2}{2\pi^2 D}\sum_{m=1}^{\infty}\frac{E_m}{m^2\cosh\alpha_m}\left(\alpha_m\tanh\alpha_m\cosh\frac{m\pi y}{a} - \frac{m\pi y}{a}\sinh\frac{m\pi y}{a}\right)\sin\frac{m\pi x}{a} \qquad ...\text{eqn. 7.5}$$

Particular Cases:

If moment is uniform along edges $y = \pm b/2$, $f_1(x) = M_0$,

$$M_0 = \sum_{m=1,3,...}^{\infty}\frac{4M_0}{m\pi}\sin\frac{m\pi x}{a}$$

i.e. $$E_m = \frac{4M_0}{m\pi}$$

$\therefore$ $$w = \frac{2M_0 a^2}{\pi^3 D}\sum_{m=1,3,...}^{\infty}\frac{1}{m^3\cosh\alpha_m}\left(\alpha_m\tanh\alpha_m\cosh\frac{m\pi y}{a} - \frac{m\pi y}{a}\sinh\frac{m\pi y}{a}\right)\sin\frac{m\pi x}{a}$$

Along middle line *i.e.* at $y = 0$,

$$w|_{y=0} = \frac{2M_0 a^2}{\pi^3 D}\sum_{m=1,3,...}^{\infty}\frac{1}{m^3\cosh\alpha_m}\cdot\alpha_m\tanh\alpha_m\sin\frac{m\pi x}{a}$$

If *a* is very large compared to *b*,

$$\tanh \alpha_m = \alpha_m \text{ and } \cosh \alpha_m = 1.$$

$$w|_{y=0} = \frac{2M_0 a^2}{\pi^3 D} \sum \frac{1}{m^3} \cdot \alpha_m^2 \sinh \frac{m\pi x}{a}$$

$$= \frac{2M_0 a^2}{\pi^3 D} \sum \frac{1}{m^3} \cdot \frac{m^2 \pi^2 b^2}{4a^2} \sin \frac{m\pi x}{a}$$

$$= \frac{M_0 b^2}{2\pi D} \sum \frac{1}{m} \sin \frac{m\pi x}{a}$$

(*b*) *Anti-symmetric Case*

Consider the anti-symmetric case shown in Fig. 7.4 in which

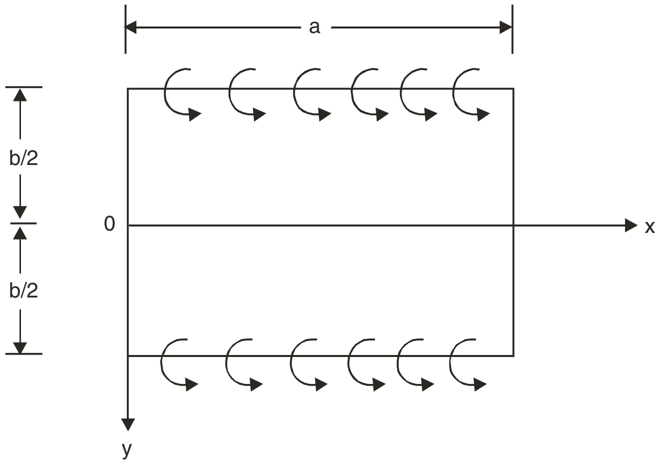


Fig. 7.4 Plate with anti-symmetric edge moment $\Sigma E'_m \sin \frac{m\pi x}{a}$

$$f_1(x) = -f_2(x) = \sum E'_m \sin \frac{m\pi x}{a}$$

Since, it is anti-symmetric case, symmetric terms in general solution must vanish.

i.e. $A_m = B_m = 0.$

Hence, $$w = \sum_{m=1}^{\infty} \left(C_m \sinh \frac{m\pi y}{a} + D_m \frac{m\pi y}{a} \cosh \frac{m\pi y}{a} \right) \sin \frac{m\pi x}{a}$$

From boundary condition, $w|_{y=\pm b/2} = 0,$ we get

$$C_m \sin \frac{m\pi b}{2a} + D_m \frac{m\pi b}{2a} \cdot \cosh \frac{m\pi b}{2a} = 0$$

i.e. $$C_m \sinh \alpha_m + D_m \alpha_m \cosh \alpha_m = 0 \quad \text{...(3)}$$

Now $$\frac{\partial w}{\partial x} = \sum_{m=1}^{\infty} \left[\frac{m\pi}{a} \left(C_m \cosh \frac{m\pi y}{a} + D_m \cosh \frac{m\pi y}{a} + D_m \frac{m\pi y}{a} \cdot \sinh \frac{m\pi y}{a} \right) \right] \sin \frac{m\pi x}{a}$$

$\therefore$ $$\frac{\partial^2 w}{\partial x^2} = \sum_{m=1}^{\infty} \frac{m^2 \pi^2}{a^2} \left[\left(C_m + D_m \right) \sinh \frac{m\pi y}{a} + D_m \sinh \frac{m\pi y}{a} + D_m \frac{m\pi y}{a} \cdot \cosh \frac{m\pi y}{a} \right] \sin \frac{m\pi x}{a}$$

$$= \sum_{m=1}^{\infty} \frac{m^2 \pi^2}{a^2} \left[\left(C_m + 2D_m \right) \sinh \frac{m\pi y}{a} + D_m \frac{m\pi y}{a} \cosh \frac{m\pi y}{a} \right] \sin \frac{m\pi x}{a}$$

From boundary condition $-D \left. \frac{\partial^2 w}{\partial y^2} \right|_{y=\pm\frac{b}{2}} = \sum E'_m \sin \frac{m\pi x}{a}$

$$-D \frac{m^2 \pi^2}{a^2} \left[\left(C_m + 2D_m \right) \sinh \frac{m\pi b}{2a} + D_m \frac{m\pi b}{2a} \cosh \frac{m\pi b}{2a} \right] = E'_m .$$

i.e. $$\left(C_m + 2D_m \right) \sinh \alpha_m + D_m \alpha_m \cosh \alpha_m = -\frac{a^2}{m^2 \pi^2 D} E'_m$$

Subtracting equation (3) from equation (4) we get

$$D_m = -\frac{a^2 E'_m}{2m^2 \pi^2 D \sinh \alpha_m}$$

Hence, from equation (3), we get

$$C_m = \frac{a^2 E'_m \alpha_m}{2m^2 \pi^2 D \sinh \alpha_m} \cdot \coth \alpha_m$$

$\therefore$ $$w = \frac{a^2}{2\pi^2 D} \sum_{m=1}^{\infty} \frac{E'_m}{m^2 \sinh \alpha_m} \left[\alpha_m \coth \alpha_m \sinh \frac{m\pi y}{a} - \frac{m\pi y}{a} \cosh \frac{m\pi y}{a} \right] \sin \frac{m\pi y}{a} \quad \text{...eqn. 7.6}$$

The following differences in equation 7.5 and 7.6 may be noted:

In symmetric case		In anti-symmetric case
$\tanh \alpha_m$	$\to$	$\coth \alpha_m$
$\cosh \alpha_m$	$\to$	$\sinh \alpha_m$
$\cosh \frac{m\pi y}{a}$	$\to$	$\sinh \frac{m\pi y}{a}$
and $\sinh \frac{m\pi y}{a}$	$\to$	$\cosh \frac{m\pi y}{a}$.

(c) *If Moments at y = –b/2 and y = b/2 are Different*

If the moments applied along edges $y = -b/2$ and $y = b/2$ are different, they may be split into a symmetric case and an anti-symmetric case as shown in Fig. 7.2. Let,

symmetric moment be $E_m \sin\frac{m\pi x}{a}$ and anti-symmetric moment be $E'_m \sin\frac{m\pi x}{a}$. Then obviously solution for such case is

$$w = \frac{a^2}{2\pi^2 D}\sum_{m=1}^{\infty}\frac{\sin\frac{m\pi x}{a}}{m^2}\left[\frac{E_m}{\cosh\alpha_m}\left(\alpha_m \tanh\alpha_m \cosh\frac{m\pi y}{a} - \frac{m\pi y}{a}\sinh\frac{m\pi y}{a}\right)\right.$$

$$\left.+\frac{E'_m}{\sinh\alpha_m}\left(\alpha_m \coth\alpha_m \sinh\frac{m\pi y}{a} - \frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\right)\right] \quad \text{...eqn. 7.7}$$

If the bending moment is acting only along the edge $y = -b/2$, moment along edge $y = b/2$ is zero. Then symmetric moment is $\frac{f_1}{2}$ and anti-symmetric moment is also $\frac{f_1}{2}$.

7.2 RECTANGULAR PLATE WITH TWO OPPOSITE EDGES SIMPLY SUPPORTED AND THE OTHER TWO EDGES FIXED

Figure 7.5 shows a typical rectangular plate considered. The plate is subjected to *udl* q_0.

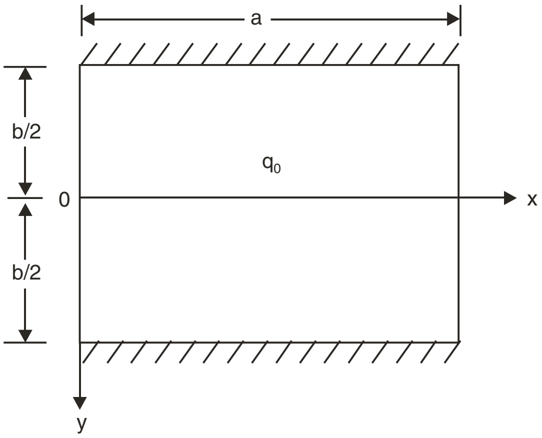


Fig. 7.5 Plate with edges $y = \pm b/2$ fixed

This problem can be solved by Levy's method applying boundary conditions at $y = \pm b/2$ as $w = 0$ and $\frac{\partial w}{\partial y} = 0.$ However, here it is solved by superposition of an appropriate edge moment case on the simply supported case. The value of moment to be applied at the edges is arrived from the consideration that when the two cases are combined slope at fixed edges should be zero.

Figure 7.6 shows how given problem can be split into two cases. The solution for case I and case II clubbed with the condition that

$$\frac{\partial w_1}{\partial y} + \frac{\partial w_2}{\partial y} = 0 \text{ at } y = \pm\frac{b}{2}$$

gives the solution for the given case. Thus, the deflection for the given case is

$$w = w_1 + w_2.$$

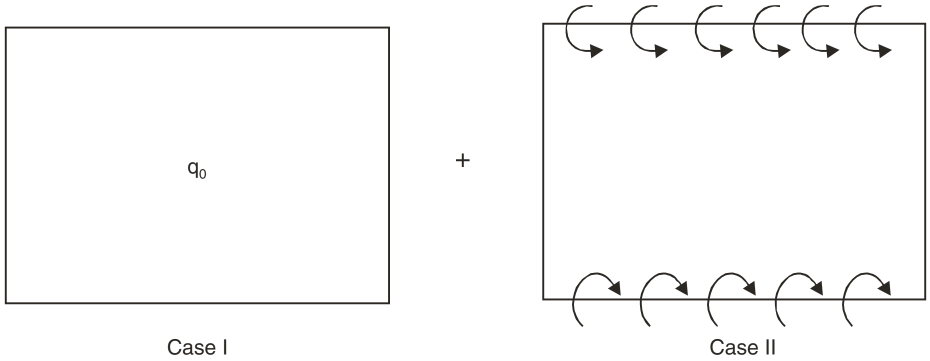


Fig. 7.6 Splitting the case shown in Figure 7.5 into two cases

From equation 6.5, we know for case I,

$$w_1 = \sum_{m=1,3\ldots}^{\infty} \frac{4q_0a^4}{m^5\pi^5 D}\left[1 - \frac{2+\alpha_m \tanh\alpha_m}{2\cosh\alpha_m}\cosh\frac{m\pi y}{a} + \frac{1}{2\cosh\alpha_m}\frac{m\pi y}{a}\sinh\frac{m\pi y}{a}\right]\sin\frac{m\pi x}{a}$$

$$\therefore \quad \frac{\partial w_1}{\partial y} = \sum_{m=1,3\ldots}^{\infty} \frac{4q_0a^4}{m^5\pi^5 D}\frac{m\pi}{a}\left[-\frac{2+\alpha_m \tanh\alpha_m}{2\cosh\alpha_m}\sinh\frac{m\pi y}{a}\right.$$

$$\left. + \frac{1}{2\cosh\alpha_m}\sinh\frac{m\pi y}{a} + \frac{1}{2\cosh\alpha_m}\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\right]\sin\frac{m\pi y}{a}$$

$$\therefore \quad \left.\frac{\partial w_1}{\partial y}\right|_{y=\pm\frac{b}{2}} = \frac{4q_0a^3}{\pi^4 D}\sum_{m=1,3\ldots}^{\infty}\frac{1}{m^4}\left[-\frac{(2+\alpha_m \tan\alpha_m)}{2}\tanh\alpha_m + \frac{1}{2}\tanh\alpha_m + \frac{1}{2}\alpha_m\right]\sin\frac{m\pi y}{a}$$

$$= \frac{2q_0a^3}{\pi^4 D}\sum_{m=1,3\ldots}^{\infty}\frac{1}{m^4}\left[\alpha_m - \tanh\alpha_m\left(1+\alpha_m \tanh\alpha_m\right)\right]\sin\frac{m\pi y}{a}$$

w_2 for this case is given by equation 7.6. Thus,

$$w_2 = \frac{a^2}{2\pi^2 D}\sum_{m=1,3\ldots}^{\infty}\frac{E_m}{m^2\cosh\alpha_m}\left[\alpha_m \tanh\alpha_m \cosh\frac{m\pi y}{a} - \frac{m\pi y}{a}\sinh\frac{m\pi y}{a}\right]\sin\frac{m\pi x}{a}$$

$$\therefore \quad \frac{\partial w_2}{\partial y} = \frac{a^2}{2\pi^2 D}\sum_{m=1,3,\ldots}^{\infty}\frac{E_m}{m^2\cosh\alpha_m}\cdot\frac{m\pi}{a}\left[\alpha_m \tanh\alpha_m \sinh\frac{m\pi y}{a}\right.$$

$$\left. - \sinh\frac{m\pi y}{a} - \frac{m\pi y}{a}\cdot\cosh\frac{m\pi y}{a}\right]\sin\frac{m\pi x}{a}$$

$$\therefore \quad \left.\frac{\partial w_2}{\partial y}\right|_{y=\pm b/2} = \frac{a}{2\pi D}\sum_{m=1,3\ldots}^{\infty}\frac{E_m}{m}\left[\alpha_m \tanh\alpha_m \tanh\alpha_m - \tanh\alpha_m - \alpha_m\right]\sin\frac{m\pi x}{a}$$

$$= \frac{a}{2\pi D}\sum_{m=1,3,\ldots}^{\infty}\frac{E_m}{m}\left[\tanh\alpha_m\left(\alpha_m \tanh\alpha_m - 1\right) - \alpha_m\right]\sin\frac{m\pi x}{a}$$

The condition to be satisfied is

$$\left.\frac{\partial w_1}{\partial y}\right|_{y=\pm\frac{b}{2}} + \left.\frac{\partial w_2}{\partial y}\right|_{y=\pm\frac{b}{2}} = 0$$

i.e.

$$\left.\frac{\partial w_1}{\partial y}\right|_{y=\pm\frac{b}{2}} = -\left.\frac{\partial w_2}{\partial y}\right|_{y=\pm\frac{b}{2}}$$

i.e.

$$\frac{2q_0a^3}{\pi^4 D}\sum_{m=1,3,\ldots}^{\infty}\frac{1}{mn}\left[\alpha_m - \tanh\alpha_m\left(1+\alpha_m\tanh\alpha_m\right)\right]\sin\frac{m\pi x}{a}$$

$$= -\frac{a}{2\pi D}\sum_{m=1,3,\ldots}^{\infty}\frac{E_m}{m}\left[\tanh\alpha_m\left(\alpha_m\tanh\alpha_m - \alpha_m\right)\right]\sin\frac{m\pi x}{a}$$

Comparing term by term and rearranging we get for all odd values of m

$$E_m = \frac{4q_0a^2}{\pi^3 m^3}\,\frac{\alpha_m - \tanh\alpha_m\left(1+\alpha_m\tanh\alpha_m\right)}{\alpha_m - \tanh\alpha_m\left(\alpha_m\tanh\alpha_m - 1\right)}$$

Hence, the bending moment along fixed edges are

$$M|_{y=\pm b/2} = \sum_{m=1,3,\ldots}^{\infty} E_m \sin\frac{m\pi x}{a}$$

$$= \frac{4q_0a^2}{\pi^3}\sum\frac{1}{m^3}\,\frac{\alpha_m - \tanh\alpha_m\left(1+\tanh\alpha_m\right)}{\alpha_m - \tanh\alpha_m\left(\alpha_m\tanh\alpha_m - 1\right)}\sin\frac{m\pi x}{a}$$

For a square plate maximum value at midspan may be obtained by substituting

$$a = b \;\; i.e.,\;\; \alpha_m = \frac{m\pi b}{2a} = \frac{m\pi}{2}.$$

and $\quad x = a/2$

$$M_{\max} = \frac{4q_0a^2}{\pi^3}\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m^3}\,\frac{\frac{m\pi}{2} - \tanh\frac{m\pi}{2}\left(1+\frac{m\pi}{2}\tanh\frac{m\pi}{2}\right)}{\frac{m\pi}{2} - \tanh\frac{m\pi}{2}\left(\frac{m\pi}{2}\tanh\frac{m\pi}{2} - 1\right)}\sin\frac{m\pi}{2}$$

Noting that $\tanh\frac{\pi}{2} = 0.9175$

$$\tanh\frac{3\pi}{2} = 0.99984$$

and for all other higher values of *m*,

$\tanh\frac{m\pi}{2} = 1.0$, we get

$$M_{\max} = \frac{4q_0 a^2}{\pi^3}\left[-0.57236 + 0.03692 - \frac{1}{5^3} + \frac{1}{7^3} - \frac{1}{9^3} + \ldots\right]$$

$$= -0.06973\ qa^2$$

Note that the first term itself gives about 6% extra moment. Thus, even if only one term of the series is considered there is only 6% overestimation.

Deflection study

Consider deflection due to edge moments only.

$$w_2 = \frac{a^2}{2\pi^2 D}\sum_{m=1,3,\ldots}^{\infty}\frac{E_m \sin\frac{m\pi x}{a}}{m^2\cosh\alpha_m}\left[\alpha_m\tanh\alpha_m\cosh\frac{m\pi y}{a} - \frac{m\pi y}{a}\sinh\frac{m\pi y}{a}\right]$$

At centre of plate $y = 0$ and $x = a/2$.

$$\therefore \qquad w_{2,\text{ centre}} = \frac{a^2}{2\pi^2 D}\sum_{m=1,3,\ldots}^{\infty}\frac{E_m \sin\frac{m\pi}{2}}{m^2\cosh\alpha_m}\left[\alpha_m\tanh\alpha_m - 0\right]$$

Substituting the value of E_m, we get

$$w_{2,\text{ centre}} = \frac{a^2}{2\pi^2 D}\sum_{m=1,3,\ldots}^{\infty}\frac{4q_0 a^2}{\pi^3 m^3}\frac{\sin\frac{m\pi}{2}}{m^2\cosh\alpha_m}\frac{\alpha_m - \tanh\alpha_m\left(1+\alpha_m\tanh\alpha_m\right)}{\alpha_m - \tanh\alpha_m\left(\alpha_m\tanh\alpha_m - 1\right)}\times\alpha_m\tanh\alpha_m$$

$$= \frac{2q_0 a^4}{\pi^5 D}\sum_{m=1,3,\ldots}^{\infty}\frac{\sin\frac{m\pi}{2}}{m^5}\frac{\alpha_m\tanh\alpha_m}{\cosh\alpha_m}\times\frac{\alpha_m - \tanh\alpha_m\left(1+\alpha_m\tanh\alpha_m\right)}{\alpha_m - \tanh\alpha_m\left(\alpha_m\tanh\alpha_m - 1\right)}$$

Noting that $\sin\frac{m\pi}{2} = (-1)^{\frac{m-1}{2}}$, we can write

$$w_{2,\text{ centre}} = \frac{2q_0 a^4}{\pi^5 D}\sum_{m=1,3,\ldots}^{\infty}\frac{(-1)^{\frac{m-1}{2}}}{m^5}\frac{\alpha_m\tanh\alpha_m}{\cosh\alpha_m}\times\frac{\alpha_m - \tanh\alpha_m\left(\alpha_m\tanh\alpha_m + 1\right)}{\alpha_m - \tanh\alpha_m\left(\alpha_m\tanh\alpha_m - 1\right)}$$

For square plate: $a = b$ *i.e.* $\alpha_m = \dfrac{m\pi}{2}$

Hence, we get

$$w_{2,\text{ centre}} = \frac{2q_0a^2}{\pi^5 D}(-0.328 + 0.000378 - ......)$$

$$= -0.00214\frac{qa^4}{D}$$

For plate with *udl,*

$$w_{1,\text{ centre}} = 0.00406\frac{q_0a^4}{D}$$

$$\therefore \qquad w = w_1 + w_2 = (0.00406 - 0.00214)\frac{q_0a^4}{D}$$

$$= 0.00192\frac{q_0a^4}{D}$$

Note that second term of the series contributes very little. Thus, it is a fast converging series. Using total solution ($w = w_1 + w_2$) it is possible to assemble all stress resultants.

7.3 RECTANGULAR PLATE WITH ALL FOUR EDGES FIXED

Figure 7.7 shows a rectangular plate with all four edges fixed and subjected to uniformly distributed load.

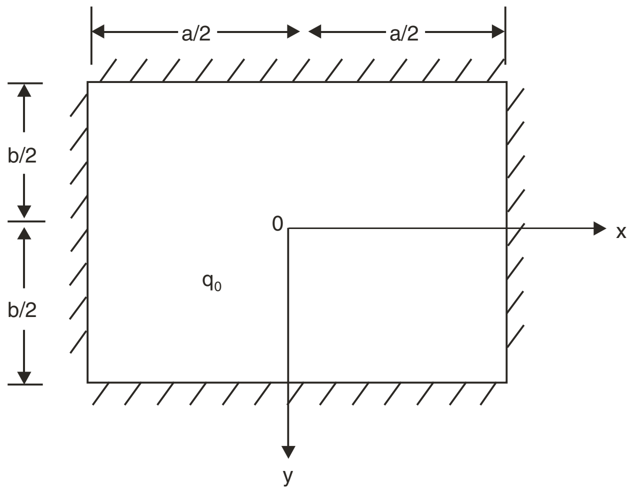


Fig. 7.7 Fixed plate subject to *udl* q_0

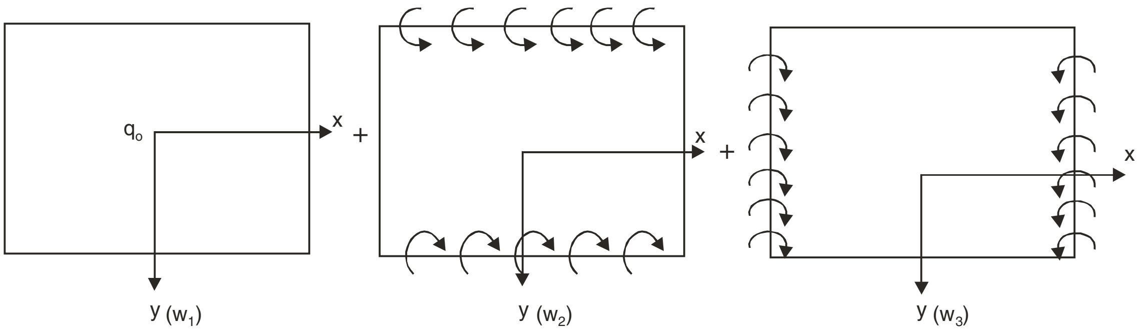


Fig. 7.8 Equivalent of fixed plate

The given case may be split into the following three cases (Refer Fig. 7.8).

Case I: A simply supported plate subject to *udl*.

Case II: A simply supported plate subject to symmetric moment along $y = \pm b/2$.

Case III: A simply supported plate subject to symmetric moment along $y = \pm a/2$.

Thus,

$$w = w_1 + w_2 + w_3.$$

For first two cases, we have already solution available. To have convenient coordinates, it is better to keep origin at centre of the plate. It may be achieved by replacing x by $x - \frac{a}{2}$ in the solutions already found for case I and case II. Then to get the solution for case III, interchange x and y in the expression for w_2. Thus, expression for w_3 is also obtained. Then the boundary conditions to be satisfied are

$$\left(\frac{\partial w_1}{\partial y} + \frac{\partial w_2}{\partial y} + \frac{\partial w_3}{\partial y} \right)_{y=\pm b/2} = 0$$

and

$$\left(\frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial x} + \frac{\partial w_3}{\partial x} \right)_{x=\pm a/2} = 0$$

From these conditions functions E_m' and E_m'' may be found.

$$w = w_1 + w_2 + w_3$$

After finding w, stress resultants may be assembled.

7.4 RECTANGULAR PLATE WITH ONE EDGE SIMPLY SUPPORTED, OTHER EDGES FIXED

The plate is *ABCD* as shown in Figure 7.9. For this first analysis may be made for a plate of size $a \times 2b$ fixed along all edges subject to anti-symmetric loading.

This case is shown in Figure 7.10. Now along edge *AB*, due to anti-symmetry

$$w = 0 \text{ and } M_y = 0.$$

Thus, it satisfies required boundary condition along *AB*. Hence, the required solution is obtained for the case considered.

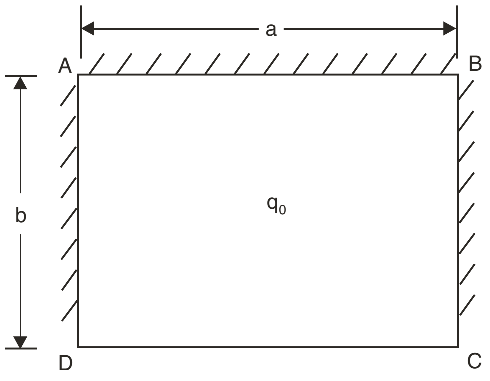


Fig. 7.9 Plate with three edges fixed and one simply supported

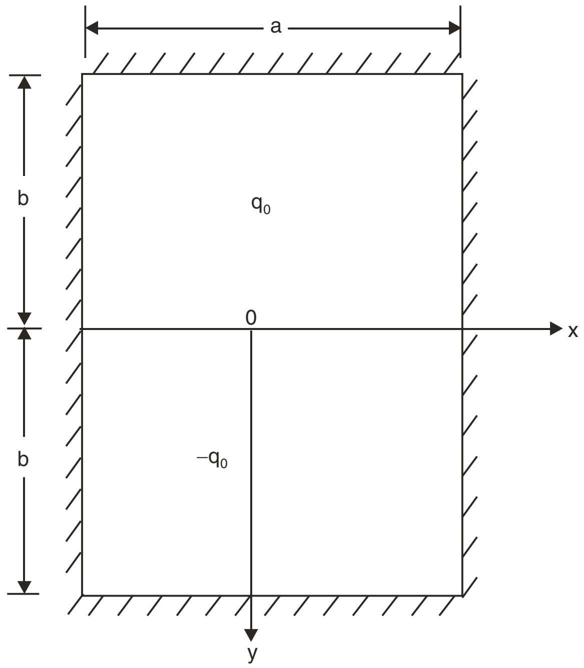


Fig. 7.10 Equivalent of plate shown in Figure 7.9

7.5 RECTANGULAR PLATE WITH TWO ADJACENT EDGES SIMPLY SUPPORTED AND OTHER TWO FIXED

This case is shown in Figure 7.11. This case may be solved from the analysis of a fixed plate subject to anti-symmetric load of size $2a \times 2b$ as shown in Figure 7.12. It satisfies the required boundary conditions along edges *AB* and *AC*.

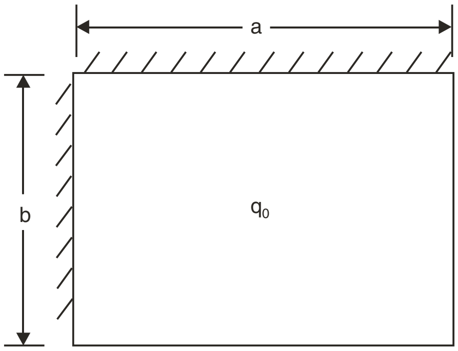


Fig. 7.11 Plate with two adjacent sides fixed and other two simply supported

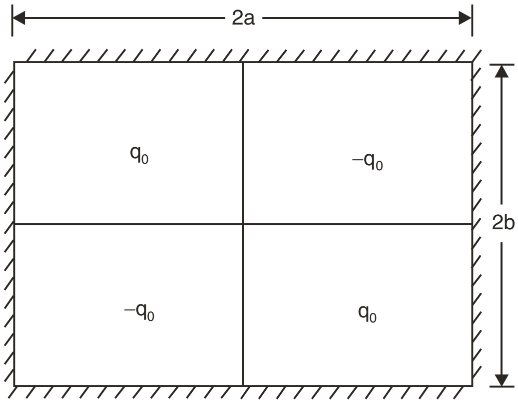


Fig. 7.12 Equivalent of plate shown in Figure 7.11

7.6 RECTANGULAR PLATE WITH TWO OPPOSITE EDGES SIMPLY SUPPORTED, ONE EDGE FIXED AND ANOTHER FREE

This case is shown in Figure 7.13.

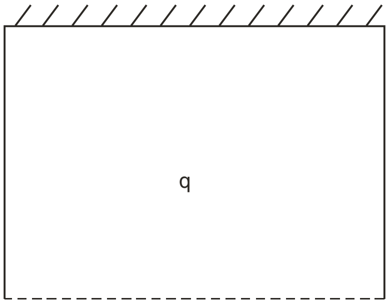


Fig. 7.13 Plate with two opposite edges simply supported, one edge fixed and other free

For this case

$$w = w_1 + w_2$$

where w_1 is particular solution and w_2 is complementary solution.

We know $$w_1 = \frac{4q_0a^4}{\pi^5 D}\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m^5}\sin\frac{m\pi x}{a}$$

and $$w_2 = \sum Y_m \sin\frac{m\pi x}{a}$$

where $$Y_m = \frac{q_0a^4}{D}\left[A_m\cosh\frac{m\pi y}{a} + B_m\frac{m\pi y}{a}\sinh\frac{m\pi y}{a} + C_m\sinh\frac{m\pi y}{a} + D_m\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\right]$$

The boundary conditions to be satisfied are

At $x = 0$ and $x = a$

$$w = 0 \text{ and } \frac{\partial^2 w}{\partial x^2} = 0$$

Hence, by taking series in the form of $\sin\frac{m\pi x}{a}$ the boundary conditions are satisfied.

The following boundary conditions help to get the constants A_m, B_m, C_m and D_m.

$$w = 0 \text{ at } y = 0$$

$$\frac{\partial w}{\partial y} = 0 \text{ at } y = 0.$$

$$\left(\mu\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = 0 \text{ at } y = b$$

$$\frac{\partial^3 w}{\partial y^3} + (2-\mu)\frac{\partial^3 w}{\partial x^2 \partial y} = 0 \text{ at } y = b$$

Hence, the total solution is obtained.

7.7 RECTANGULAR PLATES WITH THREE EDGES BUILT IN AND FOURTH EDGE FREE

Figure 7.14 shows this case. Such cases commonly appear in the design of water tanks and counterfort retaining walls.

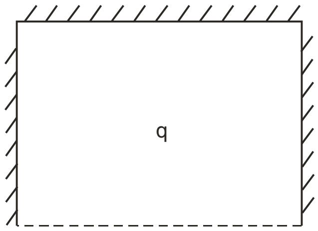


Fig. 7.14 Plate with three edges fixed, fourth free

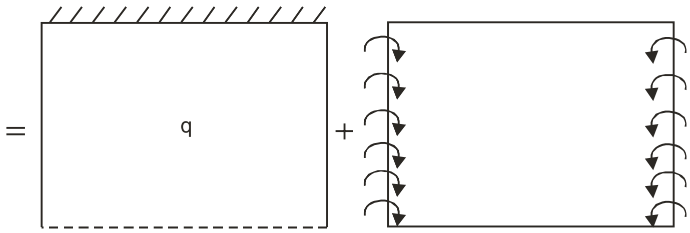


Fig. 7.15 Equivalent for plate shown in Figure 7.14

The analysis may be treated as the analysis of a plate of size $a \times b$ with three edges fixed and fourth edge free, subject to given loading (Ref. Figure 7.15) plus the solution for a plate with edges $x = 0$ and $x = a$ subject to symmetric moment with the conditions slopes at $x = 0$ or $x = a$ in x direction are zero.

7.8 RECTANGULAR PLATE CONTINUOUS IN ONE DIRECTION

Consider the analysis of continuous plate shown in Figure 7.16. The end plate 1, may be treated as a simply supported plate subject to moment $\Sigma E_{m_1} \sin \frac{m\pi x}{a}$ along the continuous edge. The intermediate plate may be treated as a simply supported plate subject to moment $\Sigma E_{m_1} \sin \frac{m\pi x}{a}$ at one edge and the moment $\Sigma E_{m_2} \sin \frac{m\pi x}{a}$ at the other edge. The third plate is treated as a simply supported plate with moment $\Sigma E_m \sin \frac{m\pi x}{a}$ at the continuous edge.

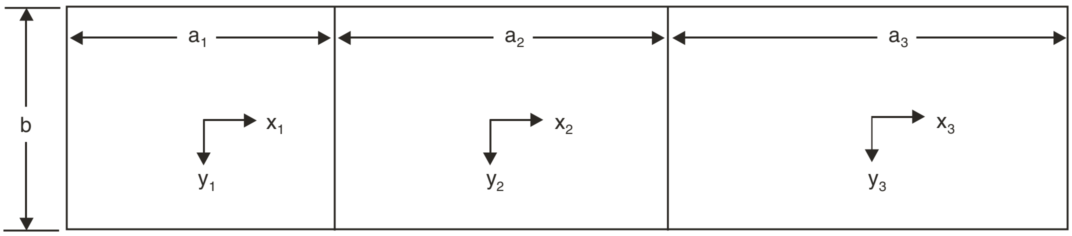


Fig. 7.16 Continuous plate in one direction

To get E_{m_1} and E_{m_2} the conditions to be satisfied are

$$\left.\frac{\partial w_1}{\partial y_1}\right|_{x_1=\frac{a_1}{2}} = \left.\frac{\partial w_2}{\partial y_2}\right|_{x_2=-\frac{a_2}{2}} \qquad ...(1)$$

and

$$\left.\frac{\partial w_2}{\partial y_2}\right|_{x_2=\frac{a_2}{2}} = \left.\frac{\partial w_3}{\partial y_3}\right|_{x_3=-\frac{a_3}{2}}.$$

It may be noted that if $a_1 = a_2 = a_3 = a$ it reduces to problem with only one unknown E_m.

One can find deflections first and then assemble coefficients for moments.

7.9 RECTANGULAR PLATES CONTINUOUS IN BOTH DIRECTIONS

This case is shown in Fig. 7.17. Slabs are usually simply supported along outer edges and continuous over beams or walls and are subjected to downward load. Any panel in the slab may be approximated as one of the six panels shown in Fig. 7.18. For these cases method of finding edge moment and stress resultants have been already discussed.

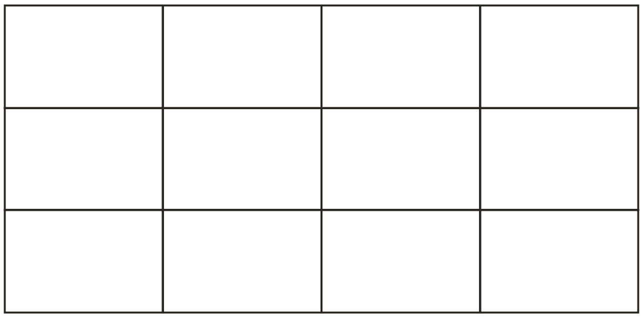


Fig. 7.17 Rectangular plate continuous in both directions

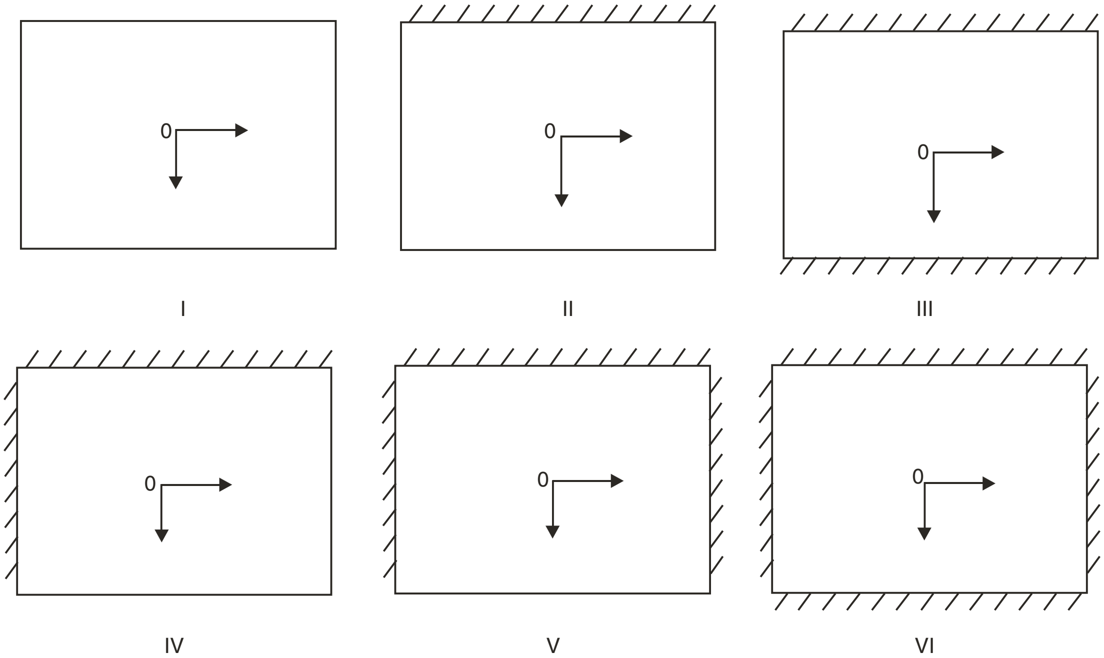


Fig. 7.18 Standard Cases

At common edges moments may slightly differ, but average moment may be approximately taken as final moment.

To get design moment at centre of the plate we know the panel should be loaded with live load and the adjacent panel should not be loaded. The solution for this case is obtained as an appropriate standard case with $q_0 + p/2$ load plus a simply supported plate with $p/2$ load where q_0 is dead load and p is live load.

The second part considered is appropriate since if we consider a load of $p/2$ applied in the checker bound form as shown in Figure 7.19, each panel behaves as a simply supported case. Hence, final moment at centre of span

= Moment at centre of span in standard case with a load of $(q_0 + p/2)$
+ Moment in simply supported slab with load of $p/2$.

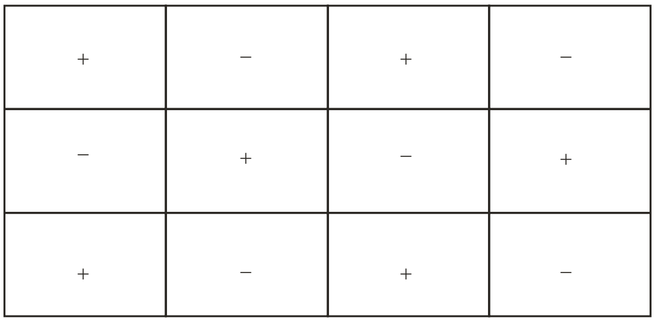


Fig. 7.19 Checker bound loading

QUESTIONS

1. Analyse a simply supported rectangular plate of size $a \times b$ if it is subjected to
   (*a*) Uniform symmetric moment at $y = \pm b/2$.
   (*b*) Uniform anti-symmetric moment at $y = \pm b/2$.
2. Discuss the extension of Levy's method for the analysis of a fixed plate subject to *udl* q_0.
3. Discuss the analysis of continuous plate to get maximum moment at mid-spans. Assume continuity is in both directions and the plate is loaded with dead load q_0 and a live load of p per unit area.

Chapter

8

Circular Plates Bent Axi-symmetrically

Circular plates are commonly used as base slabs of circular water tanks and as footings for circular columns. Usually they are subjected to uniform loads and have axi-symmetric edge conditions. Hence, the bending is axi-symmetric. In this article after deriving equation of equilibrium for axi-symmetric bending of circular plate, number of standard cases are analysed and at the end it is pointed out how several cases can be analysed by suitably combining standard cases.

8.1 CO-ORDINATES AND ELEMENT

For the analysis of circular plates polar coordinate system is more convenient. Figure 8.1 shows a typical circular plate of radius '*a*' with polar coordinates for an element of size $rd\theta \times dr$; located at distance r from centre of plate and making angle θ with a reference axis. Downward direction is positive z-coordinate direction.

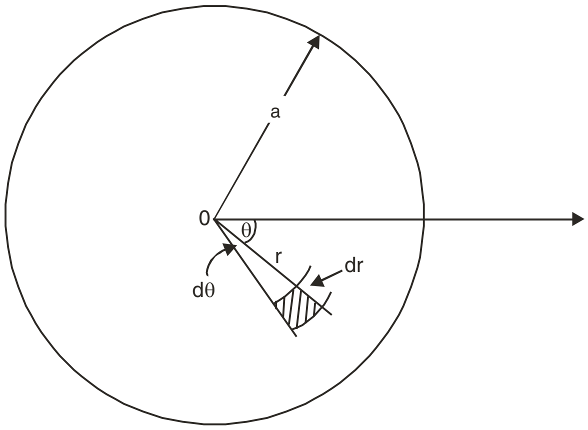


Fig. 8.1 Circular plate and polar coordinates

Figure 8.2(a) shows the moments M_r, M_θ and $M_{r\theta}$ acting on the element and Figure 8.2(b) shows the shear forces and the load intensity $q_{r\theta}$ acting on the element, where

M_r = radial moment per unit length

M_θ = Circumferential moment per unit length

$M_{r\theta} = M_{\theta r}$ = Twisting moment per unit length

Q_r = radial shear per unit length

and Q_θ = Circumferential shear per unit length.

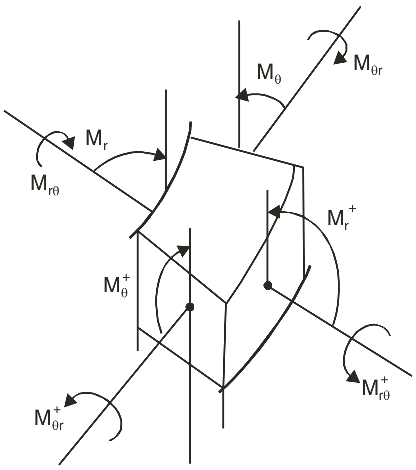


Fig. 8.2 (*a*) Moments on the element

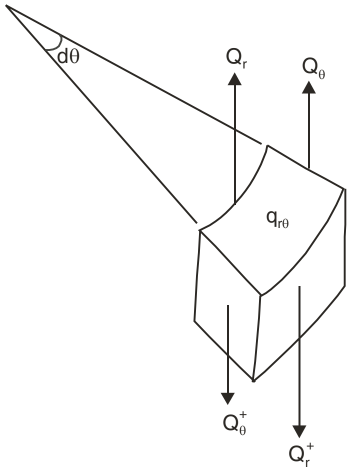


Fig. 8.2 (*b*) Forces on the element

All the stress resultants have been shown with their positive senses. The sign convention used is, a stress resultant is positive when it is acting on positive face in positive direction of coordinate or when it is acting on negative face in negative direction. It may be noted that forces on positive faces differ from these on negative faces and their relation is obviously as given below:

$$M_r^+ = M_r + \frac{\partial M_r}{\partial r} dr$$

$$M_\theta^+ = M_\theta + \frac{\partial M_\theta}{\partial \theta} \cdot d\theta$$

$$M_{\theta r}^+ = M_{\theta r} + \frac{\partial M_{\theta r}}{\partial r} \cdot dr \qquad \text{...eqn. 8.1}$$

$$M_{r\theta}^+ = M_{r\theta}^+ + \frac{\partial M_{r\theta}^+}{\partial \theta} d\theta$$

$$Q_r^+ = Q_r + \frac{\partial Q_r}{\partial r} dr$$

$$Q_\theta^+ = Q_\theta + \frac{\partial Q_\theta}{\partial \theta} \cdot d\theta$$

It may also be noted that $q_{r\theta}$ is load intensity. Hence, total down load = $q_{r\theta}\, d\theta r dr = q_{r\theta}\, r dr d\theta$

8.2 AXI-SYMMETRIC BENDING OF PLATE

If the boundary conditions and loads are axi-symmetric the plate bending also should be axi-symmetric. In other words, the condition of a strip represents the condition of the plate. Hence, in such bending twisting moment is zero and there is no variation of any force or moment with respect to θ. It means $M_{\theta r}$, $M_{r\theta}$ and Q_θ are zero. Thus in axi-symmetric bending a typical element is subjected to M_r, M_θ and Q_r only.

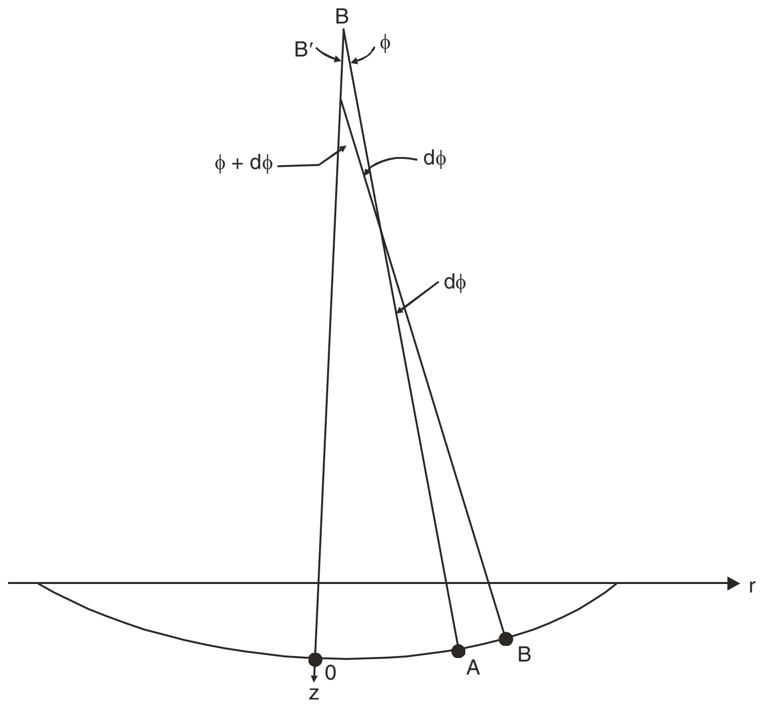


Fig. 8.3 Axi-symmetrically bent plate

Figure 8.3 shows shape of axi-symmetrically bent plate. The radii of curvatures of points at distance r meet at B and those at points $r + dr$ meet at B'. Since, bending is axi-symmetric B and B' are on a vertical line through centre of the plate. Let slope of bent plate at radius r (*i.e.* at A) be ϕ and that at $r + dr$ (*i.e.* at A') be $\phi + d\phi$. Hence, radius of curvature AB makes angle ϕ with vertical through centre O and $A'B'$ makes angle $\phi + d\phi$ with vertical through O as shown in the figure.

Slope at $A = -\dfrac{\partial w}{\partial r}$, since as r increases w decreases.

i.e.
$$\phi = -\frac{\partial w}{\partial r} \qquad \text{...eqn. 8.2}$$

Curvature of the middle surface in the diametral section rz is,

$$\frac{1}{AC} = \frac{1}{r_n} = \frac{d\phi}{dr} = -\frac{\partial^2 w}{\partial r^2} \qquad \text{...eqn. 8.3}$$

The second principal curvature is in the θ-direction. It may be observed that the normals such as *AB* for all points at radial distance *r* form a conical surface with *B* as apex. Hence, *AB* is the radius of principal curvature in θ-direction. Thus,

$$r_\theta = AB.$$

From Figure 8.3, it is clear that,

$$r = AB \cdot \phi = r_\theta \phi$$

∴ Second principal curvature is

$$\frac{1}{r_\theta} = \frac{\phi}{r} = -\frac{1}{r}\frac{\partial w}{\partial r} \qquad \text{...eqn. 8.4}$$

Hence, the expression for M_r is given by

$$M_r = D\left(\frac{1}{r_n} + \frac{1}{r_\theta}\mu\right)$$

$$= -D\left(\frac{\partial^2 w}{\partial r^2} + \frac{\mu}{r}\frac{\partial w}{\partial r}\right) \qquad \text{...eqn. 8.5}$$

Similarly

$$M_\theta = D\left(\frac{\mu}{r_n} + \frac{1}{r_\theta}\right)$$

$$= -D\left(\frac{1}{r}\frac{\partial w}{\partial r} + \mu\frac{\partial^2 w}{\partial r^2}\right) \qquad \text{...eqn. 8.6}$$

Equilibrium of the element

As stated earlier in axi-symmetrical bending any element is subjected to M_r, M_θ, Q_r and applied load intensity. Figure 8.4(*a*) shows the section and Figure 8.4(*b*) shows the plan of the element at (*r*, θ).

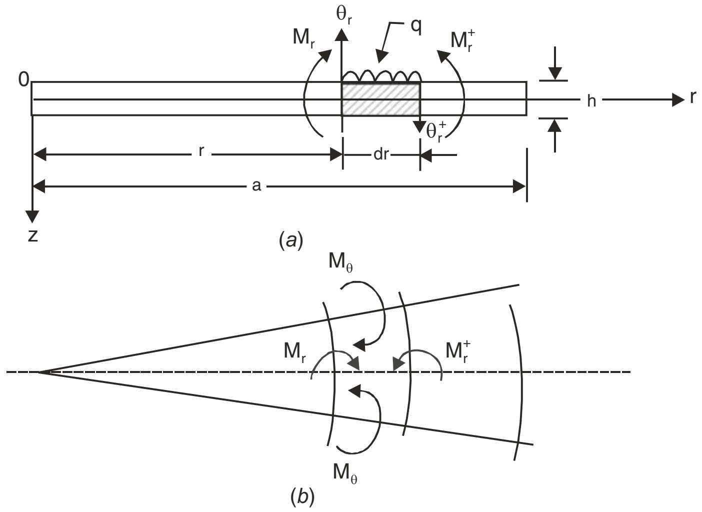


Fig. 8.4 (*a*) Sectional view (*b*) Plan view

Now,
$$M_r^+ = M_r + \frac{\partial M_r}{\partial r} dr$$

$$M_\theta^+ = M_\theta + \frac{\partial M_\theta}{\partial \theta} \cdot d\theta$$

$$= M_\theta \quad \text{since, there is no variations w.r.t. } \theta$$

$$Q_r^+ = Q_r + \frac{\partial Q_r}{\partial r} dr$$

From equilibrium condition $\sum M_r = 0$, we get

$$\left(M_r + \frac{\partial M_r}{\partial r} dr\right)(r + dr)\, d\theta - M_r r d\theta - 2\left(M_\theta dr\right)\sin\frac{d\theta}{2}$$

$$-\left(Q_r + \frac{\partial Q_r}{\partial r} dr\right)(r + dr)\, d\theta \frac{dr}{2} - Q_r r d\theta \frac{dr}{2} = 0$$

Since, $d\theta$ is small quantity we can take $\sin\frac{d\theta}{2} = \frac{d\theta}{2}$

Neglecting small quantity of higher order, we get

$$M_r dr\, d\theta + \frac{\partial M_r}{\partial r} r\, dr\, d\theta - M_\theta dr\, d\theta - Q_r r\, dr\, d\theta = 0$$

Throughout dividing by $r\, dr\, d\theta$, we get

$$\frac{\partial M_r}{\partial r} + \frac{M_r}{r} - \frac{M_\theta}{r} - Q_r = 0$$

Substituting $M_r = -D\left(\frac{\partial^2 w}{\partial r^2} + \frac{\mu}{r}\frac{\partial w}{\partial r}\right)$

and $M_\theta = -D\left(\mu \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{r}\right)$, we get

$$-D\left(\frac{\partial^2 w}{\partial r^3} + \frac{\mu}{r}\frac{\partial^2 w}{\partial r^2} - \frac{\mu}{r^2}\frac{\partial w}{\partial r}\right) - \frac{D}{r}\left(\frac{\partial^2 w}{\partial r^2} + \frac{\mu}{r}\frac{\partial w}{\partial r}\right) - \frac{1}{r}(-D)\left(\mu \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) = Q_r$$

$$\frac{\partial^3 w}{\partial r^3} + \frac{\mu}{r}\frac{\partial^2 w}{\partial r^2} - \frac{\mu}{r^2}\frac{\partial w}{\partial r} + \frac{1}{r}\frac{\partial^2 w}{\partial r^2} + \frac{\mu}{r^2}\frac{\partial w}{\partial r} - \mu\frac{\partial^2 w}{\partial r^2} - \frac{1}{r^2}\frac{\partial w}{\partial r} = -\frac{Q_r}{D}$$

$$\frac{\partial^3 w}{\partial r^3} + \frac{1}{r}\frac{\partial^2 w}{\partial r^2} - \frac{1}{r^2}\frac{\partial w}{\partial r} = -\frac{Q_r}{D}$$

i.e.
$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)\right] = -\frac{Q_r}{D} \qquad \text{...eqn. 8.7}$$

The above equation is known as plate equation for axisymmetric plate bending.

Example 8.1. Analyse a circular plate of radius '*a*' supported throughout at its outer edge and subjected to uniform moment *M*.

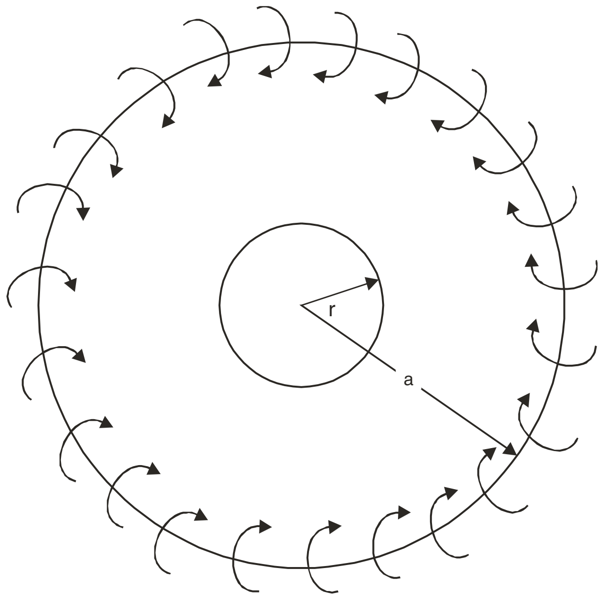


Fig. 8.5 Example 8.1

Solution. The plate equation is

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)\right] = -\frac{Q_r}{D}$$

In this case, $Q_r = 0$.

$$\therefore \qquad \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)\right] = 0.$$

$$\therefore \qquad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) = C_1, \text{ where } C_1 \text{ is a constant.}$$

i.e.

$$\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) = C_1 r$$

Integrating both sides again, we get

$$r\frac{\partial w}{\partial r} = C_1\frac{r^2}{2} + C_2, \text{ where } C_2 \text{ is a constant.}$$

$$\therefore \qquad \frac{\partial w}{\partial r} = \frac{C_1 r}{2} + \frac{C_2}{r}$$

Integrating both sides once again, we get $w = \frac{C_1 r^2}{4} + C_2 \log\frac{r}{a} + C_3,$ where C_3 is a constant. Note that in the above form instead of $C_2 \log r$ it is written as $C_2 \log r/a$, since,

$$C_2 \log r/a = C_2 \log r - C_2 \log a$$

and $C_2 \log a$ is a constant. Hence, it just alters the arbitrary constant C_3 which is yet to be determined. The form log *r/a* is convenient to apply boundary conditions.

To determine the three arbitrary constants C_1, C_2 and C_3 the following three boundary conditions are available.

$$\frac{\partial w}{\partial r} = 0 \text{ at } r = 0 \qquad ...(1)$$

$$w = 0 \text{ at } r = a \qquad ...(2)$$

and

$$M_r = M \text{ at } r = a. \qquad ...(3)$$

From boundary condition 1, we get

$$0 = \frac{C_2}{0}$$

This is possible if and only if $C_2 = 0$

$$\therefore \quad C_2 = 0$$

From boundary condition (2), we get

$$0 = \frac{C_1 a^2}{4} + C_3, \text{ since } C_2 = 0.$$

$$\therefore \quad C_3 = -\frac{C_1 a^2}{4}$$

Hence,

$$w = \frac{C_1 r^2}{4} - \frac{C_1 a^2}{4}$$

$$= -\frac{C_1}{4}\left(a^2 - r^2\right)$$

From boundary condition (3),

$$M_r\Big|_{r=a} = M$$

$$-D\left(\frac{\partial^2 w}{\partial r^2} + \frac{\mu}{r}\frac{\partial w}{\partial r}\right)\Bigg|_{r=a} = M$$

$$D\frac{C_1}{4}\left[-2 + \frac{\mu}{r}(-2r)\right] = M$$

$$-\frac{DC_1}{2}\left[1+\mu\right] = M$$

$$\therefore \quad C_1 = -\frac{2M}{D(1+\mu)}$$

$\therefore$
$$w = \frac{2M}{D(1+\mu)} \times \frac{1}{4}\left(a^2 - r^2\right)$$

i.e.
$$w = \frac{M}{2D(1+\mu)}\left(a^2 - r^2\right) \qquad \text{...eqn. 8.8}$$

$\therefore$ At any point,

$$M_r = -D\left(\frac{\partial^2 w}{\partial r^2} + \frac{\mu}{r}\frac{\partial w}{\partial r}\right)$$

$$= -D\left[\frac{M}{2D(1+\mu)}(-2) + \frac{\mu}{r}\frac{M(-2r)}{2D(1+\mu)}\right]$$

$$= \frac{M}{1+\mu}[1+\mu]$$

$$= M$$

Similarly,
$$M_\theta = \frac{M}{1+\mu}(\mu + 1) = M.$$

Thus in this case at any point

$$M_r = M_\theta = M. \qquad \text{...eqn. 8.9}$$

Example 8.2. Analyse a circular plate of radius '*a*' carrying *udl q,* if its outer edge is having fixed support.

Solution. Figure 8.6 shows such plate.

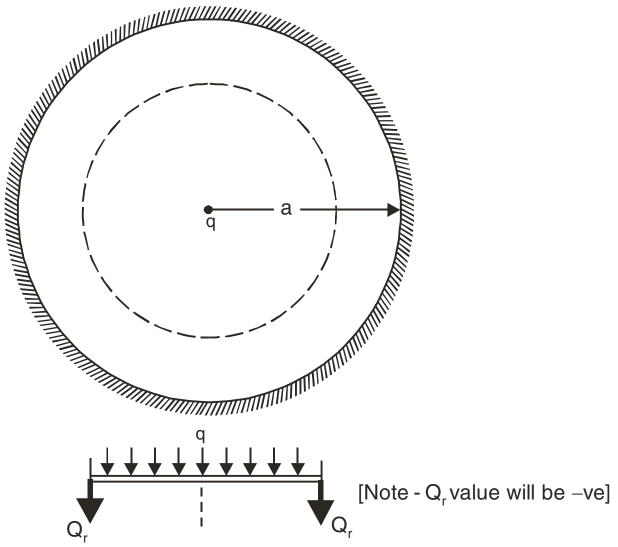


Fig. 8.6 Shear is fixed plate subject to *udl 'q'*

The shearing stress Q_r is given by

$$2\pi r\, Q_r + \pi r^2 q = 0$$

$$Q_r = -\frac{qr}{2} \quad \text{[**Note:** It is upward at outer edge.]}$$

∴ From plate equation, we get

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)\right] = \frac{-Q_r}{D} = \frac{qr}{2D}$$

∴ $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) = \frac{qr^2}{4D} + C_1$$

i.e. $$\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) = \frac{qr^3}{4D} + C_1 r$$

∴ $$r\frac{\partial w}{\partial r} = \frac{qr^4}{16D} + \frac{C_1 r^2}{2} + C_2$$

i.e. $$\frac{\partial w}{\partial r} = \frac{qr^3}{16D} + \frac{C_1 r}{2} + \frac{C_2}{r}$$

∴ $$w = \frac{qr^4}{64D} + \frac{C_1 r^2}{4} + C_2 \log r/a + C_3$$

Boundary conditions to be satisfied are,

$$\frac{\partial w}{\partial r} = 0 \text{ at } r = 0 \qquad \text{...(1)}$$

$$\frac{\partial w}{\partial r} = 0 \text{ at } r = a \qquad \text{...(2)}$$

and $$w = 0 \text{ at } r = a \qquad \text{...(3)}$$

From boundary condition 1, we get

$$0 = \frac{C_2}{0}$$

This is possible only when $C_2 = 0$.

From boundary condition 2,

$$0 = \frac{qa^3}{16D} + \frac{C_1 a}{2}, \qquad \text{since } C_2 = 0.$$

∴ $$C_1 = -\frac{qa^2}{8D}$$

∴ $$w = \frac{qr^4}{64D} - \frac{qa^2}{8D}\frac{r^2}{4} + C_3$$

Hence, from boundary condition (3),

$$0 = \frac{qa^4}{64D} - \frac{qa^4}{32D} + C_3$$

$\therefore$ $$C_3 = \frac{qa^4}{64D}$$

Thus, $$w = \frac{qr^4}{64D} - \frac{qa^2r^2}{32D} + \frac{qa^4}{64D}$$

i.e. $$w = \frac{q}{64D}\left(r^2 - a^2\right)^2 \qquad \text{...eqn. 8.10}$$

$\therefore$ Maximum deflection is at centre of the plate, where $r = 0$.

$$w_{\max} = \frac{qa^4}{64D}$$

Now, $$\frac{\partial w}{\partial r} = \frac{qr^3}{16D} - \frac{qa^2r}{16D} = \frac{qr}{16D}\left(r^2 - a^2\right)$$

and $$\frac{\partial^2 w}{\partial r^2} = \frac{3qr^2}{16D} - \frac{qa^2}{16D} = \frac{q}{16D}\left(3r^2 - a^2\right)$$

Hence,

$$M_r = -D\left(\frac{\partial^2 w}{\partial r^2} + \frac{\mu}{r}\frac{\partial w}{\partial r}\right)$$

$$= -D\left[\frac{q}{16D}\left(3r^2 - a^2\right) + \frac{\mu}{16}\frac{q}{D}\left(r^2 - a^2\right)\right]$$

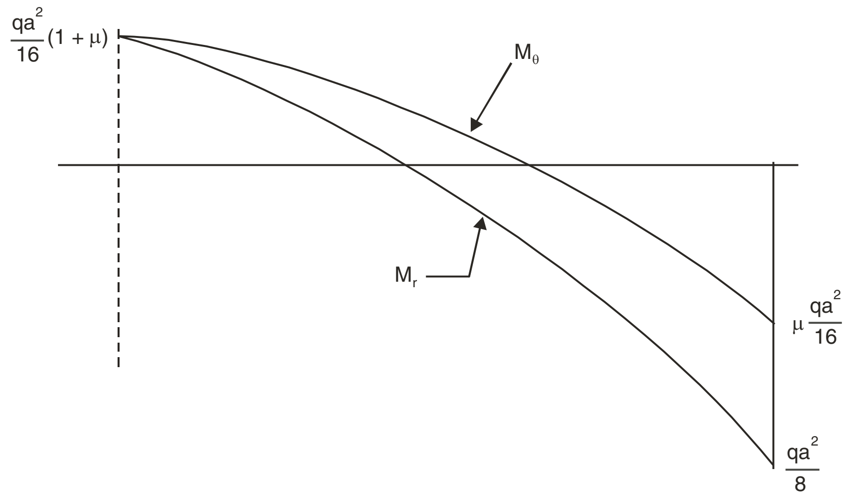


Fig. 8.7 Variation of M_r and M_θ

$$= \frac{q}{16}\left[-3r^2 + a^2 - \mu r^2 + \mu a^2\right]$$

i.e. $$M_r = \frac{q}{16}\left[a^2(1+\mu) - r^2(3+\mu)\right] \quad \text{...eqn. 8.11}$$

Similarly $$M_\theta = \frac{q}{16}\left[a^2(1+\mu) - r^2(1+3\mu)\right]$$

Variation of M_r and M_θ are as shown in Figure 8.7.

Example 8.3. Analyse a simply supported circular plate subject to *udl 'q'*.

Solution. This plate may be analysed by solving plate equation with the boundary conditions $w = 0$ and $\frac{\partial^2 w}{\partial r^2} = 0$ at $r = a,$ apart from the condition $\frac{\partial w}{\partial r} = 0$ at $r = 0.$

However, here it is solved by superposing solutions for a fixed plate subject to *udl q* and a plate supported at outer edges subjected to edge moment (Refer Figure 8.8).

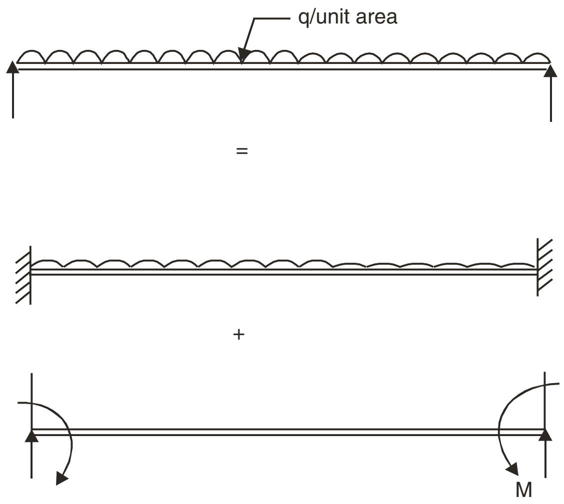


Fig. 8.8. Example 8.3

The end moment in fixed plate is $\frac{-qa^2}{8}$. If the two cases together are to represent given case, the condition to be satisfied is

$$0 = M - \frac{qa^2}{8}$$

i.e. $$M = \frac{qa^2}{8}.$$

Hence, for the given case,

$$w = w_1 + w_2$$

$$= \frac{q}{64D}\left(a^2 - r^2\right)^2 + \frac{M}{2D(1+\mu)}\left(a^2 - r^2\right)$$

$$= \frac{q}{64D}\left(a^2 - r^2\right)^2 + \frac{qa^2}{8 \times 2D(1+\mu)}\left(a^2 - r^2\right)$$

$$= \frac{q}{64D}\left(a^2 - r^2\right)\left[a^2 - r^2 + \frac{4a^2}{1+\mu}\right]$$

i.e.

$$w = \frac{q}{64D}\left(a^2 - r^2\right)\left[\frac{5+\mu}{1+\mu}a^2 - r^2\right] \qquad \text{...eqn. 8.12}$$

Similarly,

$$M_r = M_{r_1} + M_{r_2}$$

$$= \frac{q}{16}\left[a^2(1+\mu) - r^2(3+\mu)\right] + M$$

$$= \frac{q}{16}\left[a^2(1+\mu) - r^2(3+\mu)\right] + \frac{qa^2}{8}$$

$$= \frac{q}{16}\left[a^2(1+\mu+2) - r^2(3+\mu)\right]$$

$$= \frac{q}{16}(3+\mu)\left(a^2 - r^2\right) \qquad \text{...eqn. 8.13}(a)$$

$$M_\theta = M_{\theta_1} + M_{\theta_2}$$

$$= \frac{q}{16}\left[a^2(1+\mu) - r^2(1+3\mu)\right] + \frac{qa^2}{8}$$

$$= \frac{q}{16}\left[a^2(1+\mu+2) - r^2(1+3\mu)\right]$$

$$= \frac{q}{16}\left[(3+\mu)a^2 - r^2(1+3\mu)\right] \qquad \text{...eqn. 8.13}(b)$$

Note:

1. If $\mu = 0$,
   w_{centre} *in simply supported plate*

   $$= \frac{q}{64D}a^2 \times 5$$

   = 5 *times deflection in a fixed plate.*
2. $M_{r,\ \text{centre}}$ *in simply supported plate*

   $$= \frac{3}{16}qa^2$$

   = 3 × $M_{r,\ \text{centre}}$ *in fixed plate.*

Example 8.4. A circular plate of radius *'a'* is having a concentric hole of radius b and is simply supported along its outer periphery. Analyse the plate if it is subjected to

(*a*) Uniform moments M_1 and M_2 as shown in Figure 8.9.
(*b*) Uniform load Q_0 along its inner edge.

Solution.

(*a*) **Uniform Moments M_1 and M_2**

Moment M_1 is acting along inner periphery and M_2 along the outer periphery as shown in Figure 8.9.

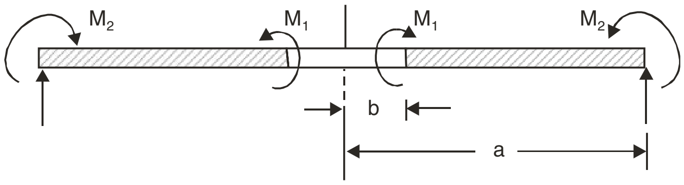


Fig. 8.9 Example 8.4(*a*)

In this case, there is no load. Hence, $Q_r = 0$

Hence, plate equation reduces to

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)\right] = 0$$

$$\therefore \qquad w = \frac{C_1 r^2}{4} + C_2 \log\frac{r}{a} + C_3$$

The constants of integrations C_1, C_2 and C_3 are to be determined by using boundary conditions.

$$M_r = M_2 \quad \text{at} \quad r = a \qquad ...(1)$$

$$M_r = M_1 \quad \text{at} \quad r = b \qquad ...(2)$$

and

$$w = 0 \quad \text{at} \quad r = a \qquad ...(3)$$

Now,

$$M_r = -D\left(\frac{\partial^2 w}{\partial r^2} + \frac{\mu}{r}\frac{\partial w}{\partial r}\right)$$

$$\frac{\partial w}{\partial r} = \frac{C_1 r}{2} + \frac{C_2}{r}$$

and

$$\frac{\partial^2 w}{\partial r^2} = \frac{C_1}{2} - \frac{C_2}{r^2}$$

$$\therefore \qquad M_r = -D\left[\frac{C_1}{2} - \frac{C_2}{r^2} + \frac{\mu C_1}{2} + \frac{\mu C_2}{r^2}\right]$$

i.e.

$$M_r = -D\left[\frac{C_1}{2}(1+\mu) - \frac{C_2}{r^2}(1-\mu)\right]$$

From boundary condition (1), we get

$$M_2 = -D\left[\frac{C_1}{2}(1+\mu) - \frac{C_2}{a^2}(1-\mu)\right] \quad ...(4)$$

From boundary condition (2),

$$M_1 = -D\left[\frac{C_1}{2}(1+\mu) - \frac{C_2}{b^2}(1-\mu)\right] \quad ...(5)$$

$\therefore$

$$M_2 - M_1 = -DC_2(1-\mu)\left[\frac{1}{b^2} - \frac{1}{a^2}\right]$$

$$= -DC_2(1-\mu)\left[\frac{a^2 - b^2}{a^2 b^2}\right]$$

$\therefore$

$$C_2 = -\frac{(M_2 - M_1)a^2 b^2}{D(1-\mu)(a^2 - b^2)}$$

Substituting it in equation (5), we get

$$M_1 = -D\left[\frac{C_1}{2}(1+\mu) + \frac{(M_2 - M_1)a^2 b^2}{D(1-\mu)(a^2 - b^2)} \times \frac{1}{b^2}(1-\mu)\right]$$

$\therefore$

$$= -D\left[\frac{C_1}{2}(1+\mu) + \frac{(M_2 - M_1)a^2}{D(a^2 - b^2)}\right]$$

$\therefore$

$$D\frac{C_1}{2}(1+\mu) = -\frac{(M_2 - M_1)a^2}{(a^2 - b^2)} - M_1$$

$$= -\left[\frac{M_2 a^2 - M_1 a^2 + M_1 a^2 - M_1 b^2}{a^2 - b^2}\right]$$

$$= -\frac{M_2 a^2 - M_1 b^2}{a^2 - b^2}$$

$$C_1 = -\frac{2(M_2 a^2 - M_1 b^2)}{D(1+\mu)(a^2 - b^2)}$$

From boundary condition (3),

$$0 = \frac{C_1 a^2}{4} + C_3$$

or

$$C_3 = -\frac{C_1 a^2}{4} = \frac{M_2 a^2 - M_1 b^2}{2D(1+\mu)(a^2 - b^2)}$$

Thus,

$$w = \frac{C_1 r^2}{4} + C_2 \log\frac{r}{a} + C_3$$

where
$$C_1 = \frac{-2\left(M_2 a^2 - M_1 b^2\right)}{D(1+\mu)\left(a^2 - b^2\right)}$$

$$C_2 = -\frac{\left(M_2 - M_1\right)a^2 b^2}{D(1-\mu)\left(a^2 - b^2\right)}$$

and
$$C_3 = \frac{M_2 a^2 - M_1 b^2}{2D(1+\mu)\left(a^2 - b^2\right)}$$

Note: If $M_2 = 0$

$$C_1 = \frac{2M_1 b^2}{D(1+\mu)\left(a^2 - b^2\right)},\ C_2 = \frac{M_1 a^2 b^2}{D(1-\mu)\left(a^2 - b^2\right)}$$

and
$$C_3 = -\frac{M_1 a^2 b^2}{2D(1+\mu)\left(a^2 - b^2\right)}$$

$\therefore$
$$w = \frac{2M_1 b^2}{D(1+\mu)\left(a^2 - b^2\right)} \times \frac{r^2}{4} + \frac{M_1 a^2 b^2}{D(1-\mu)a^2 b^2}\log\frac{r}{a} - \frac{M_1 a^2 b^2}{2D(1+\mu)\left(a^2 - b^2\right)}$$

$$= \frac{M_1 b^2\left(r^2 - a^2\right)}{2D(1+\mu)\left(a^2 - b^2\right)} + \frac{M_1 a^2 b^2}{D(1-\mu)a^2 b^2}\log\frac{r}{a}$$

i.e.
$$w = -\frac{M_1 b^2\left(a^2 - r^2\right)}{2D(1+\mu)\left(a^2 - b^2\right)} + \frac{M_1 a^2 b^2}{D(1-\mu)a^2 b^2}\log\frac{r}{a} \qquad \text{...eqn. 8.14}$$

$$\frac{\partial w}{\partial r} = \frac{M_1 a^2 b^2}{D(1-\mu)\left(a^2 - b^2\right)}\left[\frac{r}{a^2}\frac{1-\mu}{1+\mu} + \frac{1}{r}\right] \qquad \text{...eqn. 8.15}$$

(*b*) **Uniform Load Q_0 Along Inner Edge**

Figure 8.10 shows this case

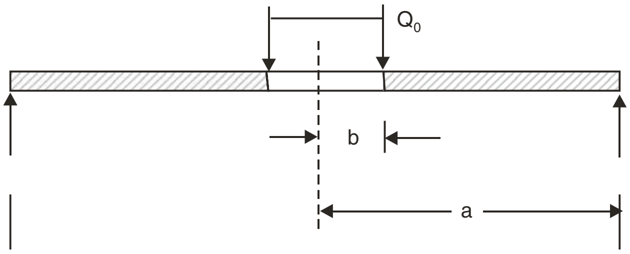


Fig. 8.10 Example 8.4(*b*)

Solution: If P is total load applied,

$$P = 2\pi b Q_0$$

At radius r shearing force per unit length of circumference is

$$Q_r = -\frac{P}{2\pi r}$$

$$\therefore \quad \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)\right] = \frac{P}{2\pi r D}$$

$$\therefore \quad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) = \frac{P}{2\pi D}\log\frac{r}{a} + C_1$$

i.e.

$$\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) = \frac{Pr}{2\pi D}\log\frac{r}{a} + C_1 r$$

$$\therefore \quad r\frac{\partial w}{\partial r} = \frac{P}{2\pi D}\left[\frac{r^2}{2}\log\frac{r}{a} - \int\frac{r^2}{2}\times\frac{1}{r}dr\right] + \frac{C_1 r^2}{2} + C_2$$

$$= \frac{P}{2\pi D}\left[\frac{r^2}{2}\log\frac{r}{a} - \frac{r^2}{4}\right] + \frac{C_1 r^2}{2} + C_2$$

$$\therefore \quad \frac{\partial w}{\partial r} = \frac{P}{2\pi D}\left[\frac{r}{2}\log\frac{r}{a} - \frac{r}{4}\right] + \frac{C_1 r}{2} + \frac{C_2}{r}$$

$$\therefore \quad w = \frac{P}{2\pi D}\left[\frac{r^2}{4}\log\frac{r}{a} - \int\frac{r^2}{4}\frac{1}{r}dr - \frac{r^2}{8}\right] + \frac{C_1 r^2}{2} + C_2\log\frac{r}{a} + C_3$$

$$= \frac{P}{2\pi D}\left[\frac{r^2}{4}\log\frac{r}{a} - \frac{r^2}{8} - \frac{r^2}{8}\right] + \frac{C_1 r^2}{2} + C_2\log\frac{r}{a} + C_3$$

$$= \frac{Pr^2}{8\pi D}\left[\log\frac{r}{a} - 1\right] + \frac{C_1 r^2}{4} + C_2\log\frac{r}{a} + C_3$$

The boundary conditions to be satisfied are

$$M_r = 0 \quad \text{at} \quad r = b \qquad \text{...(1)}$$

$$M_r = 0 \quad \text{at} \quad r = a \qquad \text{...(2)}$$

and

$$w = 0 \quad \text{at} \quad r = a. \qquad \text{...(3)}$$

Now,

$$M_r = -D\left(\frac{\partial^2 w}{\partial r^2} + \frac{\mu}{r}\frac{\partial w}{\partial r}\right)$$

$$\therefore \quad \frac{M_r}{-D} = \frac{P}{2\pi D}\left[\frac{1}{2}\log\frac{r}{a} + \frac{1}{2} - \frac{1}{4}\right] + \frac{C_1}{2} - \frac{C_2}{r^2} + \frac{\mu P}{2\pi D}\left[\frac{1}{2}\log\frac{r}{a} - \frac{1}{4}\right] + \frac{\mu C_1}{2} + \frac{\mu C_2}{r^2}$$

From boundary condition (1), we get

$$0=\frac{P}{4\pi D}\left(\log\frac{b}{a}+\frac{1}{2}\right)+\frac{\mu P}{4\pi D}\left(\log\frac{b}{a}-\frac{1}{2}\right)+\frac{C_1}{2}(1+\mu)-\frac{C_2}{b^2}(1-\mu) \qquad \text{...(4)}$$

From boundary condition (2),

$$0=\frac{P}{4\pi D}\times\frac{1}{2}+\frac{\mu P}{4\pi D}\left(-\frac{1}{2}\right)+\frac{C_1}{2}(1+\mu)-\frac{C_2}{a^2}(1-\mu) \qquad \text{...(5)}$$

Subtracting (5) from (4), we get

$$0=\frac{P}{4\pi D}\log\frac{b}{a}+\mu\frac{P}{4\pi D}\log\frac{b}{a}-C_2(1-\mu)\left(\frac{1}{b^2}-\frac{1}{a^2}\right)$$

$$=\frac{P}{4\pi D}(1+\mu)\log\frac{b}{a}-C_2(1-\mu)\frac{a^2-b^2}{a^2b^2}$$

$$\therefore \quad C_2=\frac{P}{4\pi D}\frac{1+\mu}{1-\mu}\frac{a^2b^2}{a^2-b^2}\log\frac{b}{a} \qquad \text{...(6)}$$

Substituting it in eqn. 4, we get

$$0=\frac{P}{4\pi D}\left(\log\frac{b}{a}+\frac{1}{2}\right)+\frac{\mu P}{4\pi D}\left(\log\frac{b}{a}-\frac{1}{2}\right)+\frac{C_1}{2}(1+\mu)-\frac{P}{4\pi D}(1+\mu)\frac{a^2}{a^2-b^2}\log\frac{b}{a}$$

$$\therefore \quad -\frac{C_1}{2}(1+\mu)=\frac{P}{4\pi D}\left[\log\frac{b}{a}(1+\mu)-\frac{(1+\mu)a^2}{a^2-b^2}\log\frac{b}{a}+\frac{1}{2}(1-\mu)\right]$$

$$=\frac{P}{4\pi D}\left[\frac{1}{2}(1-\mu)+\log\frac{b}{a}(1+\mu)\frac{a^2-b^2-a^2}{(a^2-b^2)}\right]$$

$$\therefore \quad C_1=-\frac{P}{4\pi D}\left[\frac{1-\mu}{1+\mu}-\frac{2b^2}{a^2-b^2}\log\frac{b}{a}\right] \qquad \text{...(7)}$$

From boundary condition (3),

$$0=\frac{Pa^2}{8\pi D}(0-1)+\frac{C_1a^2}{4}+C_3$$

$$\therefore \quad C_3=-C_1\frac{a^2}{4}+\frac{Pa^2}{8\pi D}$$

$$=\frac{Pa^2}{8\pi D}\left[1+\frac{1-\mu}{1+\mu}\times\frac{1}{2}-\frac{b^2}{a^2-b^2}\log\frac{b}{a}\right]$$

Particular case

As b approaches zero, it becomes a plate with central concentrated load. In this case, $b^2\log\frac{b}{a}=0$. Hence,

$$C_1 = -\frac{P}{4\pi D}\frac{1-\mu}{1+\mu},\ C_2 = 0 \text{ and } C_3 = \frac{Pa^2}{8\pi D}\left(1+\frac{1}{2}\frac{1-\mu}{1+\mu}\right)$$

$$\therefore \quad w = \frac{Pr^2}{8\pi D}\left(\log\frac{r}{a}-1\right)-\frac{P}{4\pi D}\frac{1-\mu}{1+\mu}\frac{r^2}{4}+\frac{Pa^2}{8\pi D}\left(1+\frac{1}{2}\frac{1-\mu}{1+\mu}\right)$$

$$= \frac{P}{8\pi D}\left[r^2\log\frac{r}{a}\right]+\frac{Pr^2}{8\pi D}\left(-1-\frac{1-\mu}{1+\mu}\frac{1}{2}\right)+\frac{Pa^2}{8\pi D}\left[1+\frac{1}{2}\frac{1-\mu}{1+\mu}\right]$$

$$= \frac{P}{8\pi D}\left[r^2\log\frac{r}{a}+\left(a^2-r^2\right)\frac{2(1+\mu)+1-\mu}{2(1+\mu)}\right]$$

$$= \frac{P}{8\pi D}\left[r^2\log\frac{r}{a}+\frac{3+\mu}{2(1+\mu)}\left(a^2-r^2\right)\right]$$

8.3 CASES OF PRACTICAL IMPORTANCE

By suitable superposition of above cases many cases of practical importance can be solved. Some of these cases are shown in Figure 8.11.

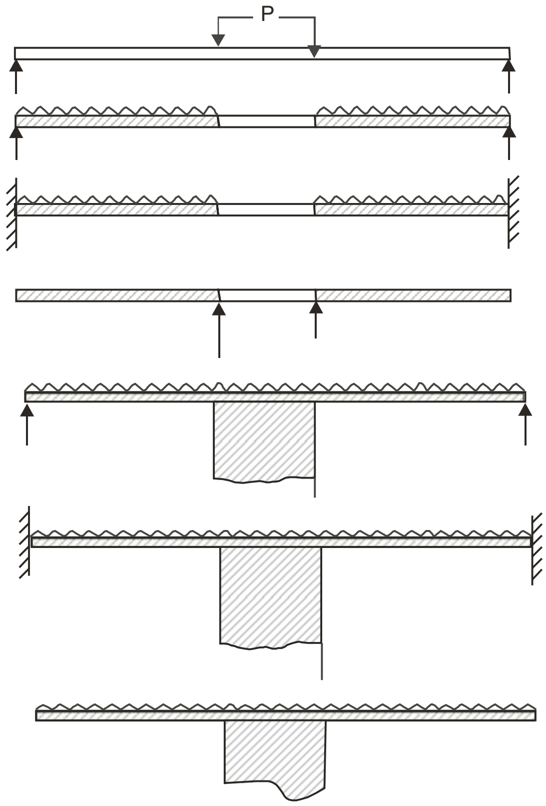


Fig. 8.11 Cases of Practical Importance

For example, consider a simply supported circular plate subjected to a concentric ring load of intensity *P* having radius *'b'* (case I).

This may be looked as combination of the following three cases:

(*i*) A simply supported plate with concentric hole subject to ring load *P*.

(*ii*) A simply supported plate with a concentric hole subjected to uniform moment M_1 along inner edge.

(*iii*) A circular plate of radius *'b'* subject to uniform moment M_1 along outer edge.

The condition to be satisfied is

$$\frac{\partial w_1}{\partial r} + \frac{\partial w_2}{\partial r} = \frac{\partial w_3}{\partial r} \text{ at } r = b.$$

The above condition helps in finding M_1.

The above three cases are shown in Figure 8.12.

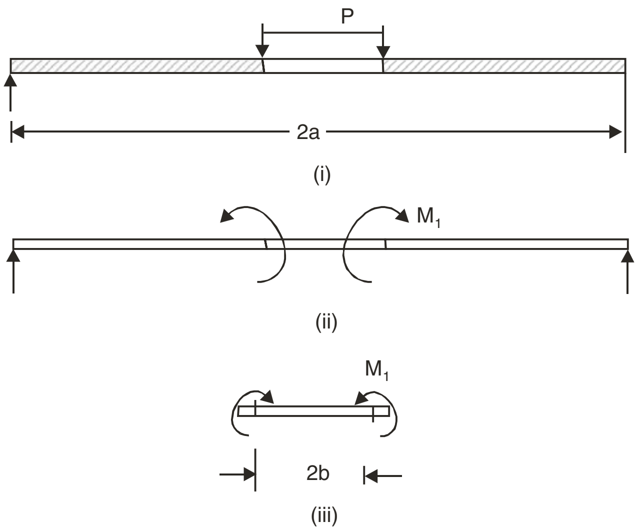


Fig. 8.12

QUESTIONS

1. Derive the equation of equilibrium for the axi-symmetrically bent circular plate.
2. Discuss the method of analysing a circular plate subject to a ring load *P*. Assume outer edge of the plate is fixed.

Chapter

9

Plates of Other Shapes

The solution for a plate of any shape can be obtained, if a deflection function in single unknown can be found to satisfy the boundary conditions. Then using the plate equation, it is possible to find the unknown parameter. In this chapter, the analysis of the following shaped plates is presented:

1. Elliptic Plates with Fixed Edges subject to
   (*a*) *udl* q_0.
   (*b*) Linearly varying load
2. Equilateral Triangular Plate subject to
   (*a*) Pure moment along simply supported edge
   (*b*) *udl* q_0.

9.1 ELLIPTIC PLATE WITH CLAMPED EDGES AND SUBJECTED TO *UDL*

Figure 9.1 shows the typical elliptic plate fixed along its outer periphery and subjected to load intensity q_0. Let its length along major axis be $2a$ and along minor axis $2b$. With the origin of cartesian coordinates at centre of plate, the equation of the boundary line is

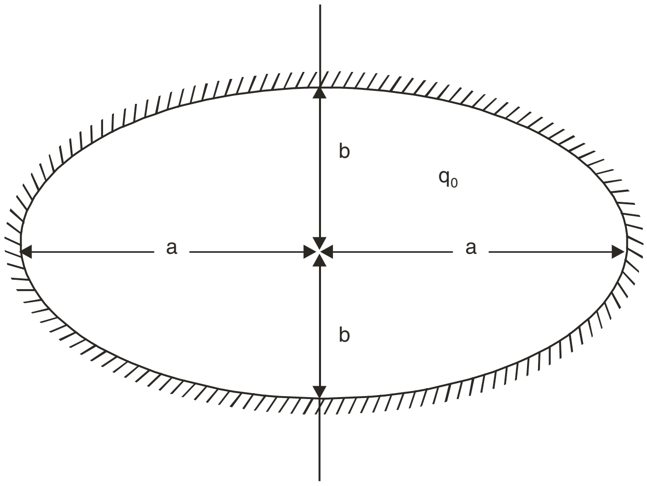


Fig. 9.1 Elliptic plate with clamped edges

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1.0=0 \qquad \text{...eqn. 9.1}$$

Along this boundary,

$$w = 0$$

and $$\frac{\partial w}{\partial n}=0. \qquad \text{...(1)}$$

i.e. $$\frac{\partial w}{\partial x}\cos\theta+\frac{\partial w}{\partial y}\sin\theta=0 \qquad \text{...(2)}$$

The deflection function should be such that the boundary conditions 1 and 2 are satisfied.

To satisfy the boundary condition $w = 0$ along the edge it is necessary that w should have a term $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)$ as a multiplying factor. The second boundary condition shows that even after first differentiation w.r.t. x and y we should be left with a term $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)$. Hence, the deflection function 'w', should contain the term $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)^2$ instead only $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)$.

Apart from satisfying boundary conditions, the following plate equation also should be satisfied.

i.e. $$\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}=\frac{q}{D} \qquad \text{...eqn. 9.2}$$

The above equation shows that, the deflection function w should not contain any term of degree higher than four so that left hand side also becomes constant as right hand side is. So, the power of term $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)$ should not be more than two. Hence, let

$$w=C\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)^2 \qquad \text{...eqn. 9.3}$$

It satisfies the boundary condition that $w = 0$ at any point on the boundary. Now

$$\frac{\partial w}{\partial n}=\frac{\partial w}{\partial x}\cos\theta+\frac{\partial w}{\partial y}\sin\theta$$

$$=2C\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\frac{2x}{a^2}\cos\theta+2C\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\frac{2y}{b^2}\sin\theta$$

Since, at any point on the boundary $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$, the boundary condition $\frac{\partial w}{\partial x}=0$ is satisfied.

From plate equation,

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}$$

Now,

$$\frac{\partial w}{\partial x} = 2C\left[\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right]\frac{2x}{a^2}$$

$$\therefore \quad \frac{\partial^2 w}{\partial x^2} = \frac{4C}{a^2}\left[\frac{3x^2}{a^2} + \frac{y^2}{b^2} - 1\right]$$

$$\frac{\partial^3 w}{\partial x^3} = \frac{4C \times 3 \times 2x}{a^4} = \frac{24x}{a^4}$$

$$\frac{\partial^4 w}{\partial x^4} = \frac{24}{a^4}$$

Similarly,

$$\frac{\partial^4 w}{\partial y^4} = \frac{24}{b^4}$$

and

$$\frac{\partial^4 w}{\partial x^2 \partial y^2} = \frac{4C}{a^2}\left(\frac{2}{b^2}\right) = \frac{8C}{a^2 b^2}$$

Hence, from plate equation,

$$C\left[\frac{24}{a^4} + \frac{2 \times 8}{a^2 b^2} + \frac{24}{b^4}\right] = \frac{q}{D}$$

or

$$C = \frac{q}{D} \quad \frac{1}{\frac{24}{a^4} + \frac{16}{a^2 b^2} + \frac{24}{b^4}}$$

Thus,

$$w = \frac{q_0}{D\left(\frac{24}{a^4} + \frac{16}{a^2 b^2} + \frac{24}{b^4}\right)}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)^2$$

or

$$w = \frac{q_0 a^4}{D\left(24 + 16\frac{a^2}{b^2} + 24\frac{a^4}{b^4}\right)}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)^2 \qquad \text{...eqn. 9.4}$$

$$\therefore \qquad w_{\text{centre}} = \frac{q_0 a^4}{D\left(24 + 16\dfrac{a^2}{b^2} + 24\dfrac{a^4}{b^4}\right)}$$

Particular Cases

If $a = b$, it is a clamped circular plate.

$$w_{\text{centre}} = \frac{q_0 a^4}{D(24+16+24)} = \frac{q_0 a^4}{64D}.$$

Same as we got in Chapter 8.
If $a = 2b$,

$$w_{\text{centre}} = \frac{q_0 a^4}{D(24+16\times 4+24\times 16)} = \frac{q_0 a^4}{472D}$$

Expressions for Moments

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right)$$

$$= -4CD\left[\left(\frac{3x^2}{a^4} + \frac{y^2}{a^2b^2} - \frac{1}{a^2}\right) + \mu\left(\frac{x^2}{a^2b^2} + \frac{3y^2}{b^4} - \frac{1}{b^2}\right)\right] \qquad \text{..eqn. 9.5}(a)$$

$$M_y = -4CD\left[\mu\left(\frac{3x^2}{a^4} + \frac{y^2}{b^2} - \frac{1}{a^2}\right) + \left(\frac{x^2}{a^2b^2} + \frac{3y^2}{b^4} - \frac{1}{b^2}\right)\right] \qquad \text{...eqn. 9.5}(b)$$

Particular Cases

(*a*) If $a = b$, it is a circular plate. In this case $C = \dfrac{q}{D}\dfrac{1}{\dfrac{24}{a^4}+\dfrac{16}{a^4}+\dfrac{24}{a^4}} = \dfrac{qa^4}{64}$

$\therefore M$ at $x = 0$, $y = 0$

$$= -4\times\frac{qa^4}{64}\frac{D}{D}\left[-\frac{1}{a^2} + \mu\left(-\frac{1}{a^2}\right)\right]$$

$= \dfrac{qa^4}{16}(1+\mu)$, same as we got in Chapter 8.

(*b*) If $a = 2b$

$$C = \frac{q}{D}\frac{1}{\dfrac{24}{a^4} + \dfrac{16}{a^2\cdot\dfrac{a^2}{4}} + \dfrac{24}{\left(\dfrac{a}{2}\right)^4}} = \frac{q}{D}\frac{1}{472}$$

$$\therefore \qquad M_{x,\text{ centre}} = -4 \times \frac{qa^4}{472}\left[-\frac{1}{a^2} - \mu\frac{1}{(a/2)^2}\right]$$

$$= \frac{qa^2}{118}[1 + 4\mu]$$

$$M_{y,\text{ centre}} = \frac{qa^2}{118}[4 + \mu]$$

Moment at the end of major axis ($x = a,\ y = b$)

$$M_x = -4\frac{qa^4}{472}\left[\left(\frac{3}{a^2} - \frac{1}{a^2}\right) + \mu\left(\frac{1}{(b)^2} - \frac{1}{(b)^2}\right)\right]$$

$$= -\frac{qa^2}{59}.$$

Moment at the end of minor axis, *i.e.* at $x = 0,\ y = b,$

$$M_y = -4\frac{qa^4}{472}\left[\mu\left(\frac{1}{(a/2)^2} - \frac{1}{a^2}\right) + \frac{3}{b^2} - \frac{1}{b^2}\right]$$

$$= -\frac{qa^4}{118}\left[\frac{\mu(-3)}{a^2} + \frac{2}{(a/2)^2}\right]$$

$$= -\frac{qa^2}{118}[8 - 3\mu]$$

The variations of moment along major and minor axes are shown in Figure 9.2, if $\mu = 0$.

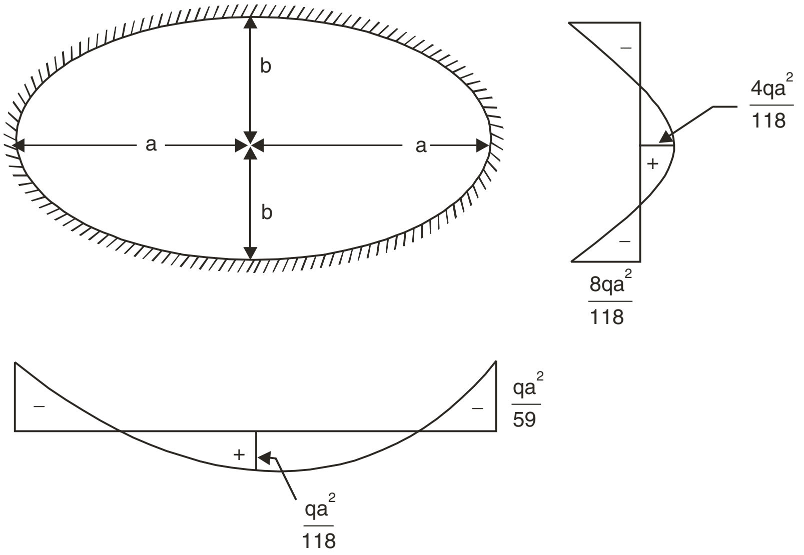


Fig. 9.2 Variations of moments along major and minor axes

9.2 COMPARISON OF RESULT WITH STRIP METHOD ANALYSIS

Let intensity of load taken by long strip be q_1 and that by short strip be q_2. Then,

$$q_1 + q_2 = q_0 \quad \text{...eqn. 9.6}$$

To ensure central deflection of the two strip same, it should be,

$$q_1 a^4 = q_2 b^4$$

$$\therefore \quad q_2 = q_1 \frac{a^4}{b^4}$$

In particular case of $a = 2b$,

$$q_2 = 16q_1$$

Substituting it in eqn. 9.5, we get

$$q_1 + 16q_1 = q_0$$

or

$$q_1 = \frac{q_0}{17} \text{ and hence } q_2 = \frac{16}{17} q_0$$

$$\therefore \quad w_{\text{centre}} = \frac{q_1 (2a)^4}{384D} = \frac{16}{17} \frac{q_0 \cdot a^4}{384D} = \frac{q_0 a^4}{408D}$$

Thus, deflection estimated in strip method is more than actual $\left(\frac{q_0 a^4}{472} \right)$. The overestimation is 15.7 percent.

Similarly central moment in case of strip method

$$= \frac{q_1 (2a)^2}{24} = \frac{q_0}{17 \times 24} \times 4a^2 = \frac{q_0 a^2}{102}$$

Actual value as found by plate theory is

$$\frac{qa^2}{118} (1 + 4\mu)$$

If μ is taken as zero, as is made in strip method, we find there is overestimation of moment in strip method to an extent $\left(\frac{118}{102} - 1 \right) \times 100 = 15.7$ percent.

9.3 ELLIPTIC PLATE SUBJECT TO UNIFORMLY VARYING LOAD IN x-DIRECTION

Figure 9.3 shows a typical fixed elliptic plate subject to a load uniformly varying load form q_0 at $x = -a$ to $-q_0$ at $x = a$.

As discussed in the act 9.1, the deflection function should contain the term $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right)^2$, so that boundary conditions are satisfied.

Loading at any point is

$$q = q_0 \frac{x}{a}.$$

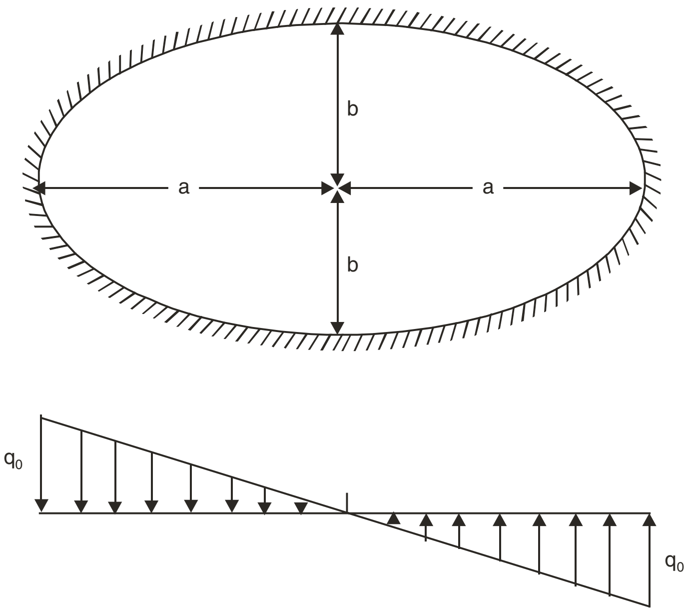


Fig. 9.3 Elliptic Plate subject to Uniformly Varying Load

Hence, to satisfy plate equation try the function

$$w = cx\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)^2$$

$$\frac{\partial w}{\partial x} = 2C\left[\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)^2 + x\times 2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\frac{2x}{a^2}\right]$$

$$\frac{\partial^2 w}{\partial x^2} = 2C\left[2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\frac{2x}{a^2}+\frac{4}{a^2}\left\{2x\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\right\}+x\frac{2x}{a^2}\right]$$

$$= \frac{4C}{a^2}\left[3x\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)+\frac{2x^2}{a^2}\right]$$

$$\frac{\partial^2 w}{\partial x^3} = \frac{4C}{a^2}\left[3\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)+3x\frac{2x}{a^2}+\frac{2}{a^2}2x\right]$$

$$= \frac{4C}{a^2}\left[3\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)+\frac{12x^2}{a^2}\right]$$

$$\frac{\partial^4 w}{\partial x^4} = \frac{4C}{a^2}\left[3\frac{2x}{a^2} + \frac{24x}{a^2}\right] = \frac{4C}{a^2} \times \frac{30x}{a^2} = \frac{120x^2}{a^4}$$

$$\frac{\partial^4 w}{\partial x^2 \partial y^2} = \frac{\partial^2}{\partial y^2}\left(\frac{\partial^2 w}{\partial x^2}\right) = \frac{4C}{a^2}\left[3x \cdot \frac{2}{b^2}\right] = \frac{24C}{a^2 b^2}$$

$$\frac{\partial w}{\partial y} = Cx\,2\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\frac{2y}{b^2}$$

$$\frac{\partial^2 w}{\partial y^2} = \frac{4C}{b^2}x\left[\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 + y\frac{2y}{b^2}\right]$$

i.e.

$$\frac{\partial^2 w}{\partial y^2} = \frac{4Cx}{b^2}\left[\frac{x^2}{a^2} + \frac{3y^2}{b^2} - 1\right]$$

$\therefore$

$$\frac{\partial^3 w}{\partial y^3} = \frac{4Cx}{b^2} \times \frac{6y}{b^2} = \frac{24Cxy}{b^4}$$

$\therefore$

$$\frac{\partial^4 w}{\partial y^4} = \frac{24Cx}{b^4}$$

$\therefore$ The plate equation is,

$$C\left[\frac{120x}{a^4} + \frac{48}{a^2 b^2}x + \frac{24x}{b^4}\right] = \frac{q_0 x}{aD}$$

$\therefore$

$$C = \frac{q_0}{aD\left[\frac{120}{a^4} + \frac{48}{a^2 b^2} + \frac{24}{b^4}\right]}$$

$\therefore$

$$w = \frac{q_0 x\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)^2}{aD\left[\frac{120}{a^4} + \frac{48}{a^2 b^2} + \frac{24}{b^4}\right]} \qquad \text{...eqn. 9.7}$$

9.4 EQUILATERAL TRIANGULAR PLATE SUBJECT TO UNIFORM EDGE MOMENT

Figure 9.4 shows the typical plate. Let the uniform edge moment be M. The origin is taken at centroid of the plate. The coordinates x and y are as shown in the figure. If we take perpendicular distance $CD = a$, the sides of equilateral triangles are $\frac{2a}{\sqrt{3}}$ (Ref. Figure 9.4).

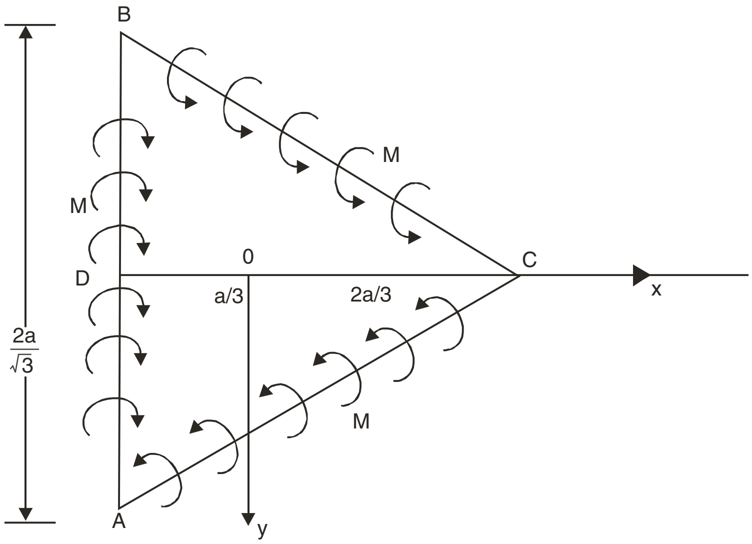


Fig. 9.4 Equilateral Triangular Plate Subject to edge moment

Equation for line *AB* is $$x + \frac{a}{3} = 0 \quad ...(1)$$

Equation for line *BC* is $$\frac{x}{2a/3} - \frac{y}{\frac{2a}{3\sqrt{3}}} - 1 = 0 \quad ...(2)$$

and equation for line *AC* is $$\frac{x}{2a/3} + \frac{y}{\frac{2a}{3\sqrt{3}}} - 1 = 0 \quad ...(3)$$

The boundary conditions to be satisfied are

$$w = 0 \quad \text{at edges} \quad ...(1)$$

and $$M_x = M \quad \text{at edges} \quad ...(2)$$

The plate equation to be satisfied is

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D} = 0, \text{ since } \frac{q}{D} = 0 \quad ...(3)$$

The plate equation indicates that no term in *w* should be of degree higher than three and boundary condition 1, indicates that expression for *w* must include the equations of boundary lines. Hence, let us try

$$w = C_1\left(x + \frac{a}{3}\right)\left(\frac{x}{2a/3} - \frac{y}{2a/3\sqrt{3}} - 1\right)\left(\frac{x}{2a/3} + \frac{y}{2a/3\sqrt{3}} - 1\right)$$

$$= C_1\left(x+\frac{a}{3}\right)\left[\left(\frac{x}{2a/3}-1\right)^2-\frac{y^2}{4a^2/27}\right]$$

$$= C_1\left(x+\frac{a}{3}\right)\left[\frac{x^2}{4a^2/9}-\frac{x}{a/3}+1-\frac{y^2}{4a^2/27}\right]$$

$$= C_1\left(x+\frac{a}{3}\right)\frac{9}{4a^2}\left[x^2-\frac{4xa}{3}+\frac{4a^2}{9}-3y^2\right]$$

$$= C_1\frac{9}{4a^2}\left[x^3-\frac{4ax^2}{3}+\frac{4a^2x}{9}-3xy^2+\frac{ax^2}{3}-\frac{4xa^2}{9}+\frac{4a^3}{27}-ay^2\right]$$

$$= \frac{9C_1}{4a^2}\left[x^3-3xy^2-ax^2-ay^2+\frac{4}{27}a^3\right]$$

$$= \frac{9C_1}{4a^2}\left[x^3-3xy^2-a\left(x^2+y^2\right)+\frac{4}{27}a^3\right]$$

$$\therefore \qquad \frac{\partial^4 w}{\partial x^4}=0, \quad \frac{\partial^4 w}{\partial x^2\partial y^2}=0 \text{ and } \frac{\partial^4 w}{\partial y^4}=0$$

Hence, the plate equation is satisfied.

$w = 0$ along the edges is satisfied, since, at any point on edges either $x+\frac{a}{3}=0$ or

$$\frac{x}{2a/3}-\frac{y}{2a/3\sqrt{3}}-1=0 \text{ or } \frac{x}{2a/3}+\frac{y}{2a/3\sqrt{3}}-1=0.$$

Another boundary condition to be satisfied is

$$M_n = M$$

i.e.

$$-D\left[\frac{\partial^2 M_n}{\partial n^2}+\mu\frac{\partial^2 M_n}{\partial t^2}\right]=M$$

Since, $\frac{\partial^2 M_n}{\partial t^2}=0$, we get $-D\frac{\partial^2 M_n}{\partial n^2}=M$

or

$$-\frac{M}{D}=\left(\frac{\partial}{\partial x}\cos\alpha+\frac{\partial}{\partial y}\sin\alpha\right)\left(\frac{\partial w}{\partial x}\cos\alpha+\frac{\partial w}{\partial y}\sin\alpha\right)$$

$$=\frac{9C}{4a^2}\left[(6x-2a)\cos^2\alpha+2(-3\times 2y)\sin\alpha\cos\alpha+(-3x\times 2-2a)\sin^2\alpha\right]$$

$$=\frac{9C}{4a^2}\left[6x\left(\cos^2\alpha-\sin^2\alpha\right)-2a\left(\cos^2\alpha+\sin^2\alpha\right)-12y\sin\alpha\cos\alpha\right]$$

$$= \frac{9C}{4a^2}[6x\cos 2\alpha - 2a - 6y\sin 2\alpha] \quad \text{...eqn. 9.8}$$

Along the boundary *AB*, *n*-direction is *x*-direction *i.e.* $\alpha = 0$ and $x = -\frac{a}{3}$.

$$\therefore \quad -\frac{M}{D} = \frac{9C}{4a^2}\left[6(-a/3) - 2a\right] = -\frac{9C}{a}$$

$$\therefore \quad C = \frac{Ma}{9C} \quad \text{...}(a)$$

Similarly along boundary *BC*, *n*-direction makes angle 60° with *x*-axis. Hence, equation 9.8 reduces to

$$-\frac{M}{D} = \frac{9C}{4a^2}\left[-3x - 2a - 6y\frac{\sqrt{3}}{2}\right]$$

$$= \frac{9C}{4a^2} \times 2a\left[-\frac{x}{2a/3} - \frac{y}{2a/3\sqrt{3}} - 1\right]$$

$$= \frac{9C \times 2a(-2)}{4a^2} \text{ since } \frac{x}{2a/3} + \frac{y}{2a/3\sqrt{3}} = 1$$

$$= -\frac{9C}{a} \quad \text{...}(b)$$

or $\quad C = \frac{Ma}{9D}$, same as in (*a*)

Similarly, for line *AC* also we get

$$C = \frac{Ma}{9D}.$$

Hence, if $C = \frac{Ma}{9D}$ the boundary condition $M_x = M$ along all edges is satisfied. Hence,

$$w = \frac{9C}{4a^2}\left[x^3 - 3xy^2 - a\left(x^2 + y^2\right) + \frac{4}{27}a^3\right]$$

i.e.

$$w = \frac{M}{4aD}\left[x^3 - 3xy^2 - a\left(x^2 + y^2\right) + \frac{4}{27}a^3\right] \quad \text{...eqn. 9.9}$$

Variation of Moments Along Line CD

$$\frac{\partial^2 w}{\partial x^2} = \frac{M}{4aD}[6x - 2a]$$

$$\frac{\partial^2 w}{\partial y^2} = \frac{M}{4aD}[-6x - 2a]$$

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{M}{4aD}[-6y]$$

$\therefore$
$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2}\right)$$

$$= -\frac{M}{4a}\left[6x - 2a + \mu(-6x - 2a)\right]$$

$$= \frac{M}{2}\left[-3(1-\mu)\frac{x}{a} + (1+\mu)\right]$$

$$M_y = \frac{M}{2}\left[3(1-\mu)\frac{x}{a} + 1 + \mu\right]$$

and
$$M_{xy} = -D(1-\mu)\frac{\partial^2 w}{\partial x \partial y}$$

$$= 3M(1-\mu)\frac{y}{2a}.$$

$\therefore$ The variation of moments along DC is as shown in Figure 9.5.

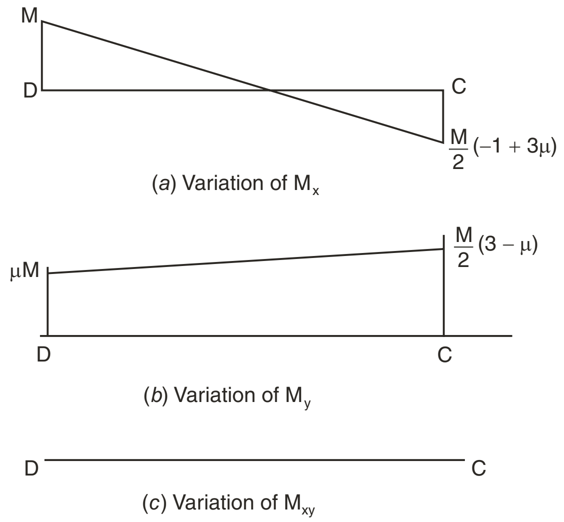


Fig. 9.5 Variation of moments along *DC*

9.5 EQUILATERAL TRIANGULAR PLATE SIMPLY SUPPORTED ALONG ITS EDGES AND SUBJECTED TO UDL

Figure 9.6 shows the case considered. In this the deflection function has to satisfy the following conditions:

$$w = 0 \text{ along all boundaries} \quad ...(1)$$

$$M_x = 0 \text{ along all boundaries} \quad ...(2)$$

and
$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = 0 \quad ...(3)$$

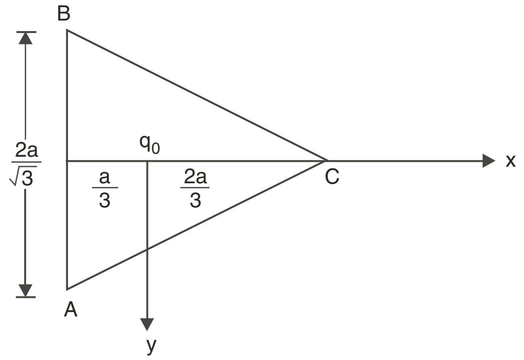


Fig. 9.6 Equilateral triangular plate subjected to UDL

From the first condition it is obvious that there should be the term

$$\left(x + \frac{a}{3}\right)\left(\frac{x}{2a/\sqrt{3}} + \frac{y}{\frac{2}{3}\frac{a}{\sqrt{3}}} - 1\right)\left(\frac{x}{2a/3} - \frac{y}{\frac{2}{3}\frac{a}{\sqrt{3}}} - 1\right)$$

i.e. a term $x^3 - 3xy^2 - a\left(x^2 + y^2\right) + \frac{4}{27}a^3$. The other two conditions suggest that there should be some more terms. Krieger suggested the following function.

$$w = C\left[x^3 - 3xy^2 - a\left(x^2 + y^2\right) + \frac{4}{27}a^3\right]\left[x^2 + y^2 - \frac{4}{9}a^2\right]$$

It may be noted that the term $x^2 + y^2 - \frac{4}{9}a^2$ is the equation of circle passing through the corners of the plate.

$M_x = 0$ means

$$\left(\frac{\partial}{\partial x}\cos\alpha + \frac{\partial}{\partial y}\sin\alpha\right)\left(\frac{\partial w}{\partial x}\cos\alpha + \frac{\partial w}{\partial y}\sin\alpha\right) = 0$$

i.e.
$$\frac{\partial^2 w}{\partial x^2}\cos^2\alpha + 2\frac{\partial^2 w}{\partial x \partial y}\cos\alpha\sin\alpha + \frac{\partial^2 w}{\partial y^2}\sin^2\alpha = 0.$$

It may be verified that $M_x = 0$ for all three sides.

Now,

$$\frac{\partial^4 w}{\partial x^4} = C[5\times4\times3\times2x - 4\times3\times2\times1a] = C[120x - 24a]$$

$$\frac{\partial^4 w}{\partial x^2 \partial y^2} = C[-3\times3\times2\times2x + 3\times2\times2x - a\times2\times2\times2]$$

$$= C[-24x - 8a]$$

$$\frac{\partial^4 w}{\partial y^4} = C[-3x\times4\times3\times2\times1 - a\times4\times3\times2\times1]$$

$$= C[-72x - 24a]$$

$\therefore$ $\nabla^4_w = q/D$ gives

$$C[120x - 24a - 2(24x + 8a) - 72x - 24a] = \frac{q}{D}$$

$$C(-64a) = \frac{q}{D}$$

or $$C = -\frac{q}{64aD}$$

$$w = \frac{q}{64D}\left[x^3 - 3xy^2 - a(x^2 + y^2) + \frac{4}{27}a^3\right]\left[\frac{4}{9}a^2 - x^2 - y^2\right] \qquad \text{...eqn. (9.10)}$$

QUESTIONS

1. Derive the expression for deflection in case of a fixed elliptic plate subject to *udl*. Show that strip method over estimates the deflection by 15.7%, if $a = 2b$.
2. Determine the expression for deflection in a fixed elliptic plate subject to a load varying linearly from q_0 at $x = -a$ to $-q_0$ at $x = a$.
3. Determine the displacement function for an equilateral plate supported along its all edges and subjected to uniform moment on the edges.

Chapter

10

Energy Method

As structure undergoes deformation due to applied load, the work done by the load is stored as strain energy. Corresponding to each type strain there is strain energy stored in the structure. The different strain energies to be considered are

1. Flexural strain energy.
2. Torsional strain energy.
3. Shear strain energy.

Shear strain energy is very small compared to other two. To make calculations simple this is neglected.

While undergoing deformation strain energy is stored while potential energy is lost. But equilibrium is reached with minimum total energy. The principal of minimum energy is used in the analysis. For this a function is assumed so as to satisfy the boundary conditions and minimization of total energy with respect to unknown parameters in the deflection function is used to determine the unknown parameters.

In this chapter, first expression for total energy is derived. Then a few plate problems are solved to illustrate how energy method can be used to analyse the plate. This method is one of the approximate methods, since, there can be a number of deflection functions which satisfy the required boundary conditions. However, any function satisfying boundary conditions gives reasonably good results.

10.1 EXPRESSION FOR TOTAL ENERGY

Figure 10.1 shows an element of plate subject to the moments M_x, M_y, M_{xy} and M_{yx}. Strain energy due to all these moments are to be assembled.

(*a*) **Strain Energy due to M_x:**

$$dv_1 = \frac{1}{2} \times \text{Moment} \times \text{Change of angle in elemental length}$$

$$= \frac{1}{2} M_x dy \times \left(-\frac{\partial^2 w}{\partial x^2} \right) dx$$

$$= -\frac{1}{2} M_x \frac{\partial^2 w}{\partial x^2} dx\, dy$$

(*b*) **Strain Energy due to M_y:**

Similary $dv_2 = -\frac{1}{2} M_y \frac{\partial^2 w}{\partial y^2} dx\, dy$

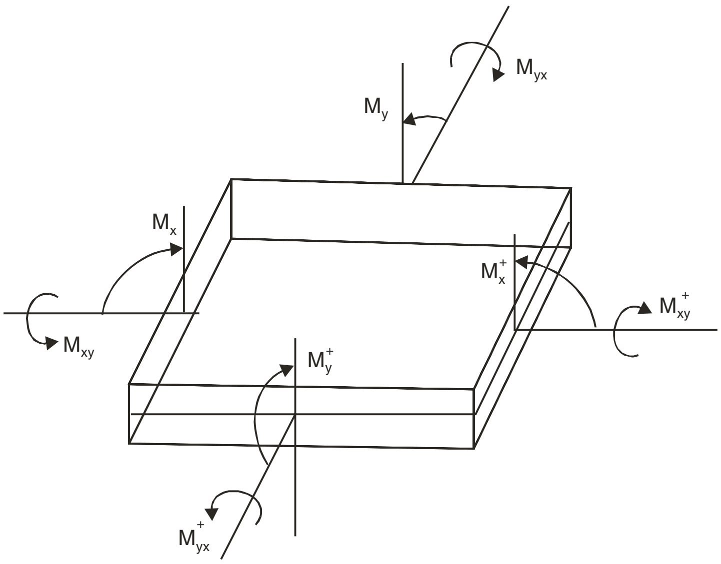


Fig. 10.1 Element of plate subject to various moments

(*c*) **Strain Energy due to M_{xy}:**

$$dv_3 = \frac{1}{2} \times \text{Twisting moment} \times \text{Change in twist}$$

$$= \frac{1}{2} M_{xy} dy \cdot \left(-\frac{\partial^2 w}{\partial x \partial y} \right) dx$$

$$= -\frac{1}{2} M_{xy} \frac{\partial^2 w}{\partial x \partial y} dx\, dy$$

(*d*) **Strain Energy due to M_{yx}:**

Similarly,

$$dv_4 = \frac{1}{2} M_{yx} dx \left(-\frac{\partial^2 w}{\partial x \partial y} dy \right)$$

$$= -\frac{1}{2} M_{yx} \frac{\partial^2 w}{\partial x \partial y} dx\, dy$$

$\therefore$ Total strain energy in the element = $dv_1 + dv_2 + dv_3 + dv_4$

$$= -\frac{1}{2} \left[M_x \frac{\partial^2 w}{\partial x^2} + M_y \frac{\partial^2 w}{\partial y^2} + M_{xy} \frac{\partial^2 w}{\partial x \partial y} + M_{yx} \frac{\partial^2 w}{\partial x \partial y} \right]$$

Since, $M_{xy} = M_{yx}$, we get

$$\text{Total S.E. in the element} = -\frac{1}{2} \left[M_x \frac{\partial^2 w}{\partial x^2} + M_y \frac{\partial^2 w}{\partial y^2} + 2 M_{xy} \frac{\partial^2 w}{\partial x \partial y} \right]$$

Substituting $$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2}\right)$$

$$M_y = -D\left(\mu \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)$$

and $$M_{xy} = -D(1-\mu)\frac{\partial^2 w}{\partial x \partial y}$$

we get

$$\text{Total S.E. in the element} = -\frac{D}{2}\left[\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + \mu\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial y^2}\cdot\frac{\partial^2 w}{\partial y^2} + 2(1-\mu)\frac{\partial^2 w}{\partial x \partial y}\cdot\frac{\partial^2 w}{\partial x \partial y}\right]$$

Representing $\frac{\partial w}{\partial x}$ by $'$

and $\frac{\partial w}{\partial y}$ by $\cdot$

and total strain energy in the element by dV, we get,

$$dV = \frac{D}{2}\left[w''^2 + \ddot{w}^2 + 2\mu w''\ddot{w} + 2(1-\mu)w'^{\cdot 2}\right]dx\,dy$$

$$= \frac{D}{2}\left[(w'' + \ddot{w})^2 - 2w''\ddot{w} + 2\mu w''\ddot{w} + 2(1-\mu)w'^{\cdot 2}\right]dx\,dy$$

$$= \frac{D}{2}\left[(w'' + \ddot{w})^2 - 2(1-\mu)(w''\ddot{w} - w'^{\cdot 2})\right]dx\,dy$$

Hence, total strain energy in the plate is

$$V = \frac{D}{2}\int\int (w'' + \ddot{w})^2 - 2(1-\mu)(w''\ddot{w} - w'^{\cdot 2})\,dx\,dy \qquad \text{...eqn. 10.1}$$

Potential energy lost may be represented as

$$U = -\int\int qw\,dx\,dy$$

∴ Total energy of the plate

$$I = U + V$$

$$= \frac{D}{2}\int\int\left[(w'' + \ddot{w})^2 - 2(1-\mu)(w''\ddot{w} - w'^{\cdot 2})\right]dx\,dy - \int\int q\cdot w\,dx\,dy \qquad \text{...eqn. 10.2}$$

Example 10.1. By energy method analyse a simply supported plate of size $a \times b$ subject to uniformly distributed load q_0 over its entire surface.

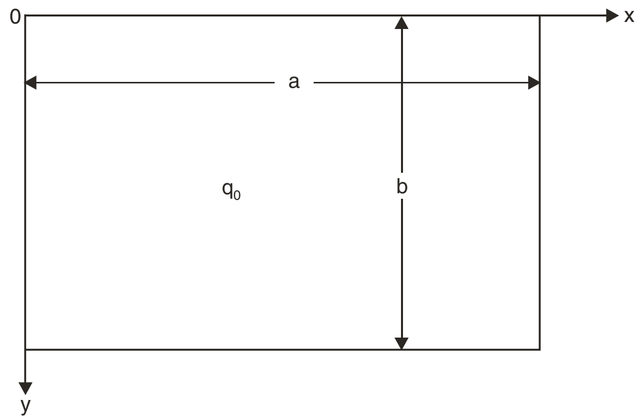


Fig. 10.2 Simply supported plate subject to *udl*

Solution. Figure 10.2 shows such plate. For such plate the following deflection function is suitable, since, the boundary conditions are easily satisfied:

$$w = \sum\sum a_{mn} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

In the above expression a_{mn} is the unknown parameter. Total energy is given by expression 8-2. First term in total strain energy:

$$\int\int(w'' + \ddot{w})^2\, dx\,dy = \int\int a_{mn}^2\left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}\right)\sin^2\frac{m\pi x}{a}\sin^2\frac{n\pi y}{b}$$

Noting the following:

$$\int_0^a \sin\frac{m\pi x}{a}dx = \frac{2a}{m\pi}, \text{ for odd values of } m$$

$$\int_0^b \sin\frac{n\pi y}{b}dy = \frac{2b}{n\pi}, \text{ for odd values of } n$$

$$\int_0^a \sin^2\frac{m\pi x}{a}dx = \frac{a}{2}$$

$$\int_0^b \sin^2\frac{n\pi y}{b}dy = \frac{b}{2}$$

$$\int_0^a \sin\frac{m\pi x}{a}\cdot\sin\frac{m'\pi x}{a}dx = 0$$

$$\int_0^b \sin\frac{n\pi y}{b}\cdot\sin\frac{n'\pi y}{b}dy = 0$$

$$\int_0^a\int_0^b \sin^2\frac{m\pi x}{a}dx\sin^2\frac{n\pi y}{b}dy=\frac{ab}{4}$$

If sin θ is replaced by cos θ in the above expressions and the limits changed from $-a/2$ to $a/2$, exactly same results are obtained.

Hence,

$$\text{First term}=\frac{D}{2}a_{mn}^2\sum\sum\left(\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}\right)^2\frac{ab}{4}$$

$$\text{Second term}=-\frac{D}{2}2(1-\mu)\int\int(w''\ddot{w})\,dx\,dy$$

$$=-D(1-\mu)\int\int a_{mn}^2\frac{m^2\pi^2}{a^2}\frac{n^2\pi^2}{b^2}\sin^2\frac{m\pi x}{a}\sin^2\frac{n\pi y}{b}dx\,dy$$

$$=-D(1-\mu)\sum\sum a_{mn}^2\frac{m^2\pi^2}{a^2}\cdot\frac{n^2\pi^2}{b^2}\frac{ab}{4}$$

Similarly third term

$$\frac{D}{2}2(1-\mu)\int\int w'^{\cdot 2}\,dx\,dy$$

$$=D(1-\mu)\sum\sum a_{mn}^2\frac{m^2\pi^2}{a^2}\frac{n^2\pi^2}{b^2}\frac{ab}{4}$$

Since, 2nd and 3rd terms are equal but opposite, strain energy is given by

$$V=\frac{\pi^4a^4ab\,D}{8}\sum\sum\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2a_{mn}^2$$

Potential energy

$$U=-\int\int q\,w\,dx\,dy$$

$$=-\int\int q\left(\sum\sum a_{mn}\sin\frac{m\pi x}{a}\cdot\sin\frac{n\pi y}{b}\right)dx\,dy$$

$$=-q\sum_{m=1,3,\ldots}^{\infty}\sum_{n=1,3,\ldots}^{\infty}a_{mn}\frac{2a}{m\pi}\cdot\frac{2b}{n\pi}$$

$$=-\frac{4qab}{\pi^2}\sum_{m=1,3,\ldots}^{\infty}\sum_{n=1,3,\ldots}^{\infty}\frac{a_{mn}}{mn}$$

∴ Total Energy

$$I=V+U$$

$$= \frac{\pi^4 a^4 ab}{8} D \sum\sum a_{mn}^2 \left(\frac{m^2}{\pi^2} + \frac{n^2}{b^2}\right)^2 - \frac{4qab}{\pi^2} \sum_{m=1,3,...}^{\infty} \sum_{n=1,3,...}^{\infty} \frac{a_m}{mn}$$

$\therefore$ $$\frac{\partial I}{\partial a_{mn}} = 0, \text{ gives}$$

For even terms a_{mn} = 0 and for odd terms

$$\frac{\pi^4 ab}{8} D\, 2a_{mn} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 - \frac{4q_0 ab}{\pi^2} \frac{1}{mn} = 0$$

$\therefore$ $$a_{mn} = \frac{16q_0}{\pi^6 D\, mn \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2} = \frac{16q_0 a^4}{\pi^6 D\, mn \left(m^2 + \frac{a^2}{b^2} n^2\right)^2}$$

This is same as Navier's solution.

Example 10.2. Analyse a fixed plate subject to uniformly distributed load q_0 (Ref. Figure 10.3)

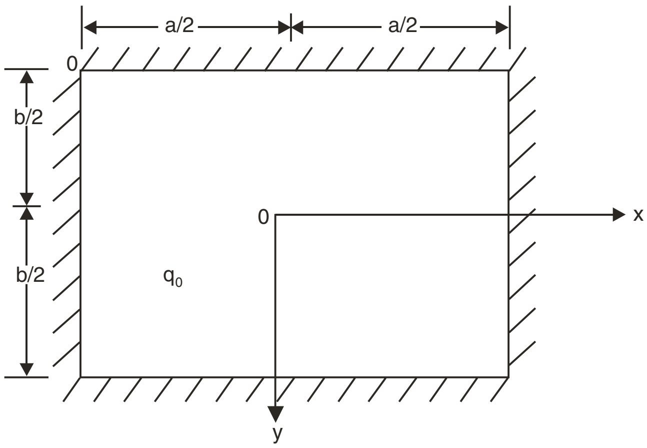


Fig. 10.3 Fixed plate subject to *udl*

Solution. In this case, the boundary conditions to be satisfied are

$$w = 0 \text{ at all edges} \quad ...(1)$$

$$\frac{\partial w}{\partial x} = 0 \text{ at } x = \frac{-a}{2} \text{ and } x = \frac{a}{2} \quad ...(2)$$

$$\frac{\partial w}{\partial y} = 0 \text{ at } y = \frac{-b}{2} \text{ and } y = \frac{b}{2} \quad ...(3)$$

Let us select

$$w = C\left[1 - \cos\frac{2\pi x}{a}\right]\left[1 + \cos\frac{2\pi y}{b}\right]$$

For simplicity series term is not selected. The function satisfies the boundary conditions $w = 0$ at all edges.

$$\frac{\partial w}{\partial x} = C\frac{2\pi}{a}\left(-\sin\frac{2\pi x}{a}\right)\left(1+\cos\frac{2\pi y}{b}\right)$$

$$\frac{\partial w}{\partial y} = C\left(1+\cos\frac{2\pi x}{a}\right)\frac{2\pi}{b}\left(-\sin\frac{2\pi x}{b}\right)$$

Hence,
$$\frac{\partial w}{\partial x} = 0 \text{ at } x = \pm\frac{a}{2}$$

and
$$\frac{\partial w}{\partial y} = 0 \text{ at } y = \pm\frac{b}{2}.$$

Thus boundary conditions (2) and (3) are also satisfied.

$$\frac{\partial^2 w}{\partial x^2} = -C\frac{4\pi^2}{a^2}\left(1+\cos\frac{2\pi y}{b}\right)\cos\frac{2\pi x}{a}$$

and
$$\frac{\partial^2 w}{\partial y^2} = -C\frac{4\pi^2}{b^2}\left(1+\cos\frac{2\pi x}{a}\right)\cos\frac{2\pi y}{b}$$

It may be observed that these curvatures are not zero at $x = \pm a/2$ and $y = \pm b/2$. Apart from these at both opposite edges curvatures are the same. Hence, moment conditions are also satisfied.

However, it may be observed that

$$\frac{\partial^3 w}{\partial x^3} = 0 \text{ at } x = \pm\frac{a}{2}$$

and
$$\frac{\partial^3 w}{\partial y^3} = 0 \text{ at } y = \pm\frac{b}{2}.$$

Thus, shear conditions are not satisfied. Still such a function gives sufficiently good results and a designer can use them.

Now,
$$w'^{\cdot} = \frac{\partial^2 w}{\partial x \partial y} = C\frac{4\pi^2}{ab}\sin\frac{2\pi x}{a}\cdot\sin\frac{2\pi y}{b}$$

First term in strain energy V

$$= \frac{D}{2}\int\int (w'' + \ddot{w})^2 \, dx\, dy$$

$$= \frac{D}{2}\int\int C^2\left[\frac{4\pi^2}{a^2}\cos\frac{2\pi x}{a}\left(1+\cos\frac{2\pi y}{b}\right) + \frac{4\pi^2}{b^2}\left(1+\cos\frac{2\pi x}{a}\right)\cos\frac{2\pi y}{b}\right]^2 dx\, dy$$

Noting that $\displaystyle\int_{-a/2}^{a/2}\int_{-b/2}^{b/2}\cos^2\frac{2\pi x}{a}\,dx\,dy=\int_{-b/2}^{b/2}\frac{a}{2}\,dy=\frac{ab}{2}$

$\displaystyle\int_{-a/2}^{a/2}\cos\frac{2\pi x}{a}\,dx=0$ since m is even.

and $\displaystyle\int_{-a/2}^{a/2}\int_{-b/2}^{b/2}\cos^2\frac{2\pi x}{a}\cos^2\frac{2\pi y}{b}\,dx\,dy=\int_{-b/2}^{b/2}\frac{a}{2}\cos^2\frac{2\pi y}{b}\,dy=\frac{ab}{4}$

First term in strain energy

$$=\frac{DC^2}{2}\left[\frac{16\pi^4}{a^4}\left(\frac{ab}{2}+0+\frac{ab}{4}\right)+\frac{16\pi^4}{b^4}\left(\frac{ab}{2}+0+\frac{ab}{4}\right)+2\frac{16\pi^4}{a^2b^2}\frac{ab}{4}\right]$$

$$=\frac{DC^2}{2}\cdot 16\pi^4\left[ab\left(\frac{3}{4}\right)\times\frac{1}{a^4}+ab\left(\frac{3}{4}\right)\times\frac{1}{b^4}+\frac{ab}{a^2b^2}\right]$$

$$=4\frac{DC^2}{ab}\pi^4\left[\frac{3}{2}\cdot\frac{b^2}{a^2}+\frac{3}{2}\frac{a^2}{b^2}+1\right]$$

$$=\frac{4DC^2\pi^4}{ab}\left[1+\frac{3}{2}\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)\right]$$

Second term in strain energy

$$-\frac{D}{2}2(1-\mu)\int_0^a\int_0^b w''\ddot{w}\,dx\,dy$$

$$=-D(1-\mu)C^2\frac{16\pi^4}{a^2b^2}\int_0^a\int_0^b\cos\frac{2\pi x}{a}\cos\frac{2\pi y}{b}\left(1+\cos\frac{2\pi y}{b}\right)\left(1+\cos\frac{2\pi x}{a}\right)dx\,dy$$

$$=-D(1-\mu)C^2\frac{16\pi^4}{a^2b^2}\frac{ab}{4}$$

Third term in strain energy

$$2(1-\mu)\frac{D}{2}\int\int w'^{\cdot 2}\,dx\,dy$$

$$=D(1-\mu)C^2\frac{16\pi^4}{a^2b^2}\int\int\sin^2\frac{2\pi x}{a}\cdot\sin^2\frac{2\pi y}{b}\,dx\,dy$$

$$= D(1-\mu)C^2 \frac{16\pi^4}{a^2b^2}\frac{a}{2}\cdot\frac{b}{2}$$

Noting that second and third terms are equal but opposite in sign, we get

$$V = \text{first term only}$$

$$= \frac{4DC^2\pi^4}{ab}\left[1+\frac{3}{2}\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)\right]$$

$$U = \text{Potential Energy}$$

$$= -\int\int q\,w\,dx\,dy$$

$$= -\int_{-a/2}^{+a/2}\int_{-b/2}^{b/2} q_0 C\left(1+\cos\frac{2\pi x}{a}\right)\left(1+\cos\frac{2\pi y}{b}\right)dx\,dy$$

$$= -q_0 Cab$$

$\therefore$ Total Energy $I = U + V$

$$= -q_0 Cab + \frac{4DC^2\pi^4}{ab}\left[1+\frac{3}{2}\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)\right]$$

$\therefore$ $\frac{\partial I}{\partial C} = 0$ gives

$$-q_0 ab + \frac{8DC\pi^4}{ab}\left[1+\frac{3}{2}\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)\right]$$

or

$$C = \frac{q_0 a^2 b^2}{8D\pi^4\left[1+\frac{3}{2}\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)\right]}$$

$\therefore$

$$w = \frac{q_0 a^2 b^2\left(1+\cos\frac{2\pi x}{a}\right)\cos\left(1+\frac{2\pi y}{b}\right)}{8DC\pi^4\left[1+\frac{3}{2}\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)\right]}$$

Maximum deflection occurs at middle of the plate *i.e.* when $x = 0$ and $y = 0$.

$\therefore$

$$w_{\max} = \frac{4q_0 a^2 b^2}{8D\pi^4\left[1+\frac{3}{2}\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)\right]}$$

For square plate $a = b$,

$$w_{max} = \frac{4q_0a^2b^2}{8D\pi^4\left[1+\frac{3}{2}(1+1)\right]} = \frac{q_0a^2b^2}{8D\pi^4}$$

$$= 0.00128\frac{q_0a^4}{D}$$

Exact value is $0.00126\frac{q_0a^4}{D}$.

Note: The following function also satisfies all boundary conditions and hence it may be tried.

$$w = C\left(\frac{x}{a}\right)^2\left(1-\frac{x}{a}\right)^2\left(\frac{y}{b}\right)^2\left(1-\frac{y}{b}\right)^2$$

in which origin is at the corner of plate. It gives $w_{max} = 0.00133\frac{q_0a^4}{D}$ for a square plate.

Example 10.3. By energy method determine expressions for w, M_x, M_y, M_{xy}, Q_x, Q_y, V_x and V_y in the plate shown in Fig. 10.4, if it is loaded with uniformly distributed load of intensity q_0.

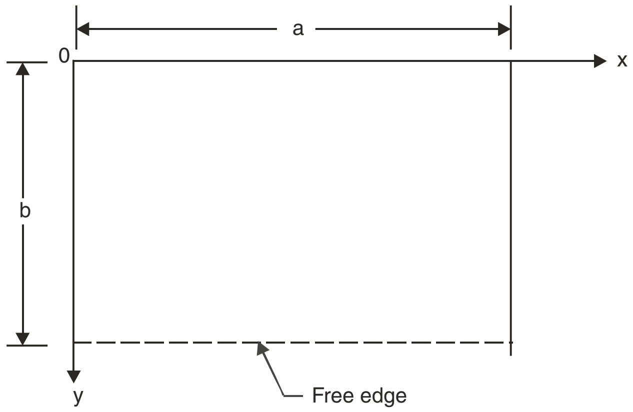


Fig. 10.4 Example 10.3

Solution. In this, the boundary conditions to be satisfied are

$$w = 0 \text{ and } \frac{\partial^2 w}{\partial x^2} = 0 \text{ at } x = 0 \text{ and } x = a \qquad ...(1)$$

$$w = 0 \text{ and } \frac{\partial^2 w}{\partial y^2} = 0 \text{ at } y = 0 \qquad ...(2)$$

$$M_y = 0 \text{ and } V_y = 0 \text{ at } y = b. \qquad ...(3)$$

These boundary conditions are satisfied if we select

$$w = Cy\sin\frac{\pi x}{a}$$

$\therefore$

$$w'' = -C\frac{\pi^2}{a^2}y\sin\frac{\pi x}{a}$$

$$\ddot{w} = 0$$

and

$$w'^{\cdot} = \frac{\partial^2 w}{\partial x \partial y} = C\frac{\pi}{a}\cos\frac{\pi x}{a}$$

$\therefore$

$$V = \frac{D}{2}\int_0^a\int_0^b\left[(w'' + \ddot{w})^2 - (w''\ddot{w} - w'^{\cdot 2})2(1-\mu)\right]dx\,dy$$

$$= \frac{D}{2}\int_0^a\int_0^b\left[\frac{C^2\pi^4}{a^4}y^2\sin^2\frac{\pi x}{a} + 2(1-\mu)C^2\frac{\pi^2}{a^2}\cos^2\frac{\pi x}{a}\right]dx\,dy$$

$$= \frac{D}{2}C^2\frac{\pi^2}{a^2}\int_0^a\int_0^b\left[\frac{\pi^2}{a^2}y^2\sin^2\frac{\pi x}{a} + 2(1-\mu)\cos^2\frac{\pi x}{a}\right]dx\,dy$$

$$= \frac{D}{2}\frac{\pi^2}{a^2}C^2\left[\frac{\pi^2}{a^2}\left\{\frac{y^3}{3}\right\}_0^b\cdot\frac{a}{2} + 2(1-\mu)\frac{ab}{2}\right]$$

$$= \frac{D}{2}\frac{\pi^2}{a^2}C^2\left[\frac{\pi^2 b^3}{3a^2}\frac{a}{2} + 2(1-\mu)\frac{ab}{2}\right]$$

$$= \frac{D}{2}\pi^2 C^2\frac{b}{a}\left[\frac{\pi^2}{6}\frac{b^2}{a^2} + (1-\mu)\right]$$

Potential Energy

$$U = -\int_0^a\int_0^b q\,w\,dx\,dy$$

$$= -\int_0^a\int_0^b q_0 Cy\sin\frac{\pi x}{a}dx\,dy$$

$$= -q_0 C\left[\frac{y^2}{2}\right]_0^b\frac{2a}{\pi}$$

$$= -q_0 C\frac{b^2 a}{\pi}$$

$$\therefore \quad I = U + V$$

$$= -q_0 C \frac{b^2 a}{\pi} + \frac{D}{2}\pi^2 C^2 \frac{b}{a}\left[\frac{\pi^2}{6}\frac{b^2}{a^2} + 1 - \mu\right]$$

$$\frac{dI}{dC} = 0, \text{ gives}$$

$$-\frac{q_0 b^2 a}{\pi} + CD\pi^2 \frac{b}{a}\left[\frac{\pi^2}{6}\frac{b^2}{a^2} + 1 - \mu\right]$$

$$\therefore \quad C = \frac{q_0 b^2 a}{\pi}\frac{a}{D\pi^2 b\left[\frac{\pi^2}{6}\frac{b^2}{a^2} + (1-\mu)\right]}$$

$$= \frac{q_0 b a^2}{D\pi^3\left[\frac{\pi^2}{6}\frac{b^2}{a^2} + (1-\mu)\right]}$$

$$\therefore \quad w = Cy \sin\frac{\pi x}{a}$$

$$= \frac{q_0 b a^2}{D\pi^3\left[\frac{\pi^2}{6}\frac{b^2}{a^2} + (1-\mu)\right]} y \sin\frac{\pi x}{a}$$

$$\therefore \quad M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right)$$

$$= DC\frac{\pi^2}{a^2} y \sin\frac{\pi x}{a}$$

$$= \frac{q_0 b}{\pi\left[\frac{\pi^2}{6}\frac{b^2}{a^2} + (1-\mu)\right]} y \sin\frac{\pi x}{a}$$

$$M_y = -D\left(\mu\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)$$

$$= -\frac{\mu q_0 b}{\pi\left[\frac{\pi^2}{6}\frac{b^2}{a^2} + (1-\mu)\right]} y \sin\frac{\pi x}{a}$$

$$M_{xy} = -D(1-\mu)\frac{\partial^2 w}{\partial x \partial y}$$

$$= -D(1-\mu)C\frac{\pi}{a}\cos\frac{\pi x}{a}$$

$$= -\frac{(1-\mu)}{\pi^2}\frac{qba}{\left[\frac{\pi^2 b^2}{6a^2}+(1-\mu)\right]}\cos\frac{\pi x}{a}$$

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}$$

$$= -\frac{q_0 by \sin\frac{\pi x}{a}}{D\pi\left[\frac{\pi^2}{6}\frac{b^2}{a^2}+(1-\mu)\right]}$$

$\therefore$

$$Q_x = -D\frac{\partial}{\partial x}(\nabla^2 w)$$

$$= \frac{q_0\frac{\pi}{a}\cdot by\cdot\cos\frac{\pi x}{a}}{\pi\left[\frac{\pi^2}{6}\frac{b^2}{a^2}+(1-\mu)\right]}$$

$$= \frac{q_0\frac{b}{a}\cdot y}{\left[\frac{\pi^2}{6}\cdot\frac{b^2}{a^2}+(1-\mu)\right]}\cdot\cos\frac{\pi x}{a}$$

$$Q_y = -D\frac{\partial}{\partial y}(\nabla^2 w)$$

$$= \frac{q_0 b}{\pi\left[\frac{\pi^2}{6}\frac{b^2}{a^2}+(1-\mu)\right]}\sin\frac{\pi x}{a}$$

$$V_x = Q_x + \frac{\partial M_{xy}}{\partial y}$$

$$= \frac{q_0\frac{b}{a}y}{\frac{\pi^2}{6}\cdot\frac{b^2}{a^2}+(1-\mu)}\cos\frac{\pi x}{a}$$

$$V_y = Q_y + \frac{\partial M_{xy}}{\partial x}$$

$$= \frac{q_0 b \cdot \sin\frac{\pi x}{a}}{\pi\left[\frac{\pi^2}{6}\cdot\frac{b^2}{a^2}+(1-\mu)\right]} + \frac{q_0 b(1-\mu)}{\pi\left[\frac{\pi^2}{6}\frac{b^2}{a^2}+(1-\mu)\right]}\sin\frac{\pi x}{a}$$

$$= \frac{q_0 b(2-\mu)}{\pi\left[\frac{\pi^2}{6}\frac{b^2}{a^2}+(1-\mu)\right]}\sin\frac{\pi x}{a}$$

QUESTIONS

1. Derive the expression for total strain energy in a plate.
2. Determine the expression for deflection in the plate shown in Figure 10.5. Use energy method.

[**Hint :** Try the function $w = c\sin\frac{\pi x}{a}\left(1-\cos\frac{\pi y}{2b}\right)$]

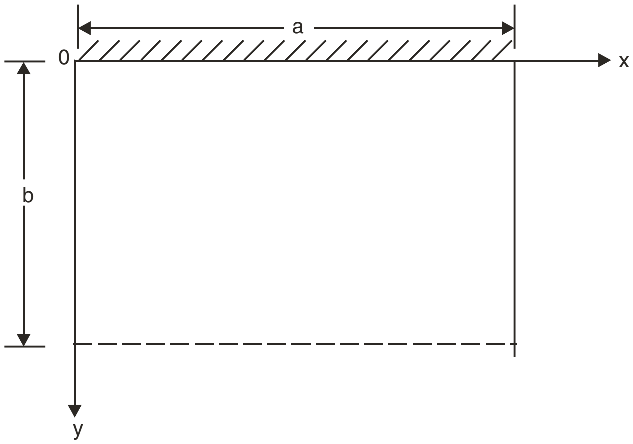


Fig. 10.5

Chapter 11

Finite Difference Method

In this method, differential equations are replaced by finite difference equations. The plate is divided into a grid. The deflection of intersection points of the grid lines are taken as unknowns. The plate equation, moments, shear etc. are expressed in terms of differences in deflections of neighbouring points. Using plate equation one equation is formed at each grid point. If any point falls outside the plate, its deflection is replaced by those of points inside for which boundary conditions are made use. The set of equations formed are solved to get deflection of each grid point.

In this chapter, first finite difference method *Vs.* classical method is discussed. Then finite difference expressions are derived for plate equations and stress resultants. After explaining how to apply boundary conditions two standard problems are solved and formation of equations for some more cases are presented.

11.1 FINITE DIFFERENCE METHOD *Vs.* CLASSICAL METHOD

1. In classical method, exact equations are formed and exact solutions are obtained, whereas in finite difference method, exact equations are formed but solved approximately.
2. Using classical method solutions are obtained for few standard cases whereas solutions can be obtained for all problems by finite difference method.
3. Whenever the complexities are encountered, classical method makes the drastic approximations and then looks for the solution. Various complexities encountered may be grouped into the following:

   (*a*) Shape
   (*b*) Boundary conditions
   (*c*) Loading.

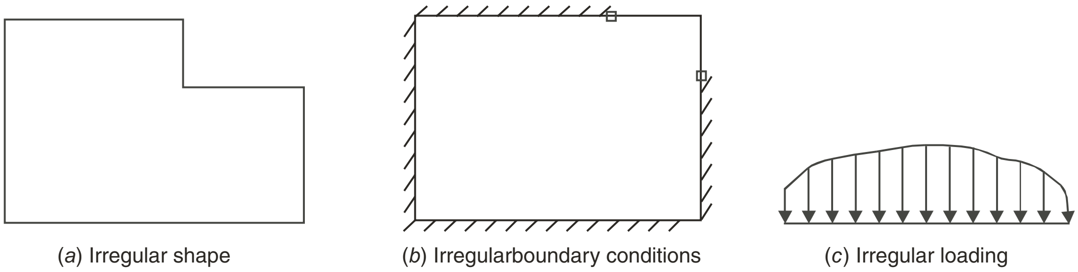


Fig. 11.1 Complexities in plate analysis

Figure 11.1 shows typical complexities in the plate analysis. To get the solution in these cases rectangular shapes, same boundary conditions along a side and regular equivalent loads are assumed. In finite difference method, no such assumptions are made. The problem is treated as it is.

4. If material property is not isotropic, solutions become difficult in classical method. Only a few simple cases have been solved successfully. Finite difference method can handle plates with anisotropic properties without any additional difficulty.

   Thus, classical method is good for standard cases while finite difference method is good for the problems with complexities in shape, boundary conditions and loading.

11.2 FINITE DIFFERENCE FORM FOR DIFFERENTIAL EQUATIONS

Let 'O' be the point under consideration (Fig. 11.2) and 'h' be the mesh length. The forward points are referred as 1, 2, 3… and the backward points as –1, –2, –3... . Thus, the deflection of point 'O' is w_0 and at points 1, 2 and 3 the defections are w_1, w_2, w_3. Similarly, deflections at points –1, –2, –3 are w_{-1}, w_{-2}, w_{-3}. The slope $\frac{\partial w}{\partial x}$ at point O is approximated as

$$\left(\frac{\partial w}{\partial x}\right)_0 = \text{Average slope at '}O\text{' towards its left and right side}$$

$$= \frac{1}{2}\left[\frac{w_0 - w_{-1}}{h} + \frac{w_1 - w_0}{h}\right]$$

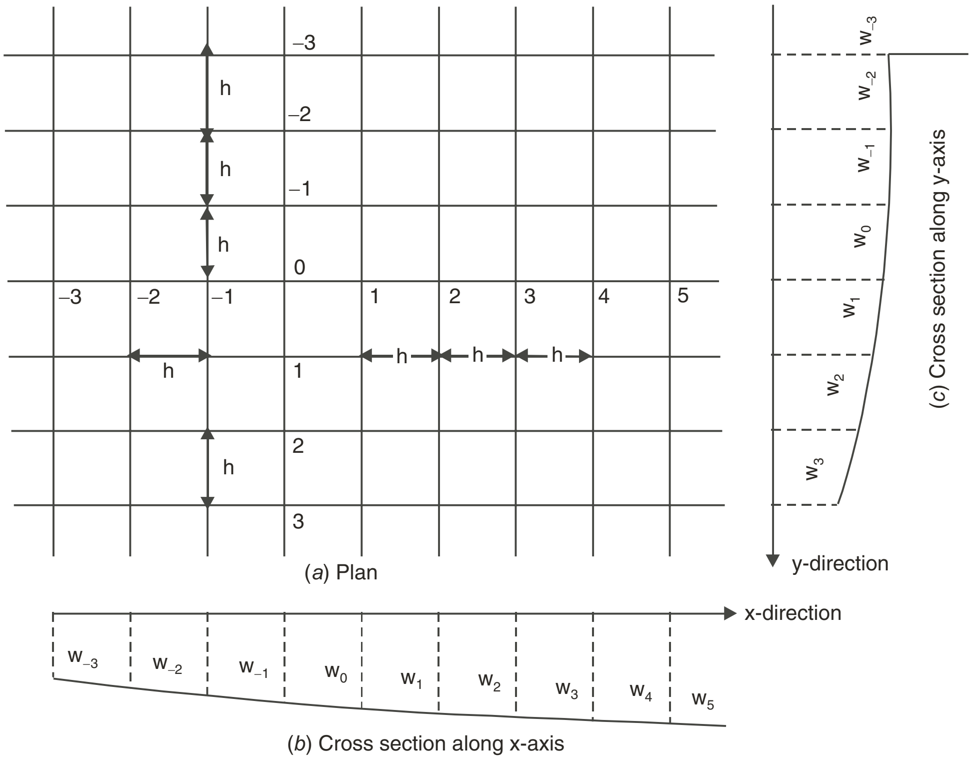


Fig. 11.2

$$= \frac{1}{2h}\left(w_1 - w_{-1}\right)$$

This finite difference form of $\frac{\partial w}{\partial x}$ may be remembered in the form

$$\left(\frac{\partial w}{\partial x}\right)_0 = \frac{1}{2h}\left[\textcircled{-1} — \circledcirc{0} — \textcircled{1}\right]_w \qquad \text{...eqn. (11.1)}$$

in which double circled point refers to the point under consideration.

Similarly, $\left(\frac{\partial^2 w}{\partial x^2}\right)$ at point '*O*' may be approximated as,

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right)$$

= rate of change of slope between points $h/2$ and $-h/2$.

$$= \frac{1}{h}\left[\frac{w_1 - w_0}{h} - \frac{w_0 - w_{-1}}{h}\right]$$

$$= \frac{w_{-1} - 2w_0 + w_1}{h^2}$$

$$= \frac{1}{h^2}\left[\textcircled{1} — \circledcirc{-2} — \textcircled{1}\right]_w \qquad \text{...eqn. (11.2)}$$

It may be noted that $\frac{\partial^2 w}{\partial x^2}$ could have been taken as rate of change of slope between the points h and $-h$ also. But in that case, as the points are far away approximation is more. Since, the slopes at middle of the range 0 to 1 and –1 to 0 could be expressed in terms of deflections at grid points, equation 11.2 is more preferred form. Similarly

$$\left(\frac{\partial^3 w}{\partial x^3}\right)_0 = \frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2}\right)$$

$$= \frac{1}{2h}\left[\left(\frac{\partial^2 w}{\partial x^2}\right)_1 - \left(\frac{\partial^2 w}{\partial x^2}\right)_{-1}\right]$$

$$\left(\frac{\partial^3 w}{\partial x^3}\right)_0 = \frac{1}{2h}\left[\frac{w_0 - 2w_1 + w_2}{h^2} - \frac{w_{-2} - 2w_{-1} + w_0}{h^2}\right]$$

$$= \frac{1}{2h^3}\left[-w_{-2} + 2w_{-1} - 2w_1 + w_2\right]$$

$$= \frac{1}{2h^3}\left[\textcircled{-1} — \textcircled{2} — \textcircled{\textcircled{0}} — \textcircled{-2} — \textcircled{1}\right]_w \qquad \text{...eqn. (11.3)}$$

The above expression may be derived conveniently by the following pattern operation also.

$$\left(\frac{\partial^3 w}{\partial x^3}\right)_0 = \frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2}\right)$$

$$= \frac{1}{2h}\left[\textcircled{-1} — \textcircled{\textcircled{0}} — \textcircled{1}\right]\frac{1}{h^2}\left[\textcircled{1} — \textcircled{\textcircled{-2}} — \textcircled{1}\right]_w$$

$$= \frac{1}{2h^3}\left[\begin{array}{l}\textcircled{-1} — \textcircled{2} — \textcircled{-1} \\ \quad + \textcircled{0} — \textcircled{\textcircled{0}} — \textcircled{0} \\ \qquad + \textcircled{1} — \textcircled{-2} — \textcircled{1}\end{array}\right]_w$$

$$= \frac{1}{2h^3}\left[\textcircled{-1} — \textcircled{2} — \textcircled{\textcircled{0}} — \textcircled{-2} — \textcircled{1}\right]_w$$

Note in the above operation, first 'O' times second pattern is placed (second row). Then −1 times second pattern is placed at one step backward (refer first row) and then 1 times second pattern is placed a step forward (refer third row).

Using pattern operation techniques $\dfrac{\partial^4 w}{\partial x^4}$ may be derived as shown below:

$$\left(\frac{\partial^4 w}{\partial x^4}\right)_0 = \frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 w}{\partial x^2}\right)$$

$$= \frac{1}{h^2}\left[\textcircled{1} — \textcircled{\textcircled{-2}} — \textcircled{1}\right]\frac{1}{h^2}\left[\textcircled{1} — \textcircled{\textcircled{-2}} — \textcircled{1}\right]_w$$

$$= \frac{1}{h^4}\left[\begin{array}{l}\textcircled{1} — \textcircled{-2} — \textcircled{1} \\ \quad + \textcircled{-2} — \textcircled{\textcircled{4}} — \textcircled{-2} \\ \qquad + \textcircled{1} — \textcircled{-2} — \textcircled{1}\end{array}\right]_w$$

$$= \frac{1}{h^4}\left[\textcircled{1} — \textcircled{-4} — \textcircled{\textcircled{6}} — \textcircled{-4} — \textcircled{1}\right]_w \qquad \text{...eqn. (11.4)}$$

It may be observed that **odd differentiations are having anti-symmetric patterns and even differentiation terms are having symmetric pattern.**

Similarly, the differentiations w.r.t. y may be expressed by turning the patterns of differentiations w.r.t. x by 90°. Thus,

$$\left(\frac{\partial w}{\partial y}\right)_0 = \frac{1}{2h}\begin{bmatrix} -1 \\ | \\ 0 \\ | \\ 1 \end{bmatrix}_w \qquad \frac{\partial^2 w}{\partial y^2} = \frac{1}{h^2}\begin{bmatrix} 1 \\ | \\ -2 \\ | \\ 1 \end{bmatrix}_w$$

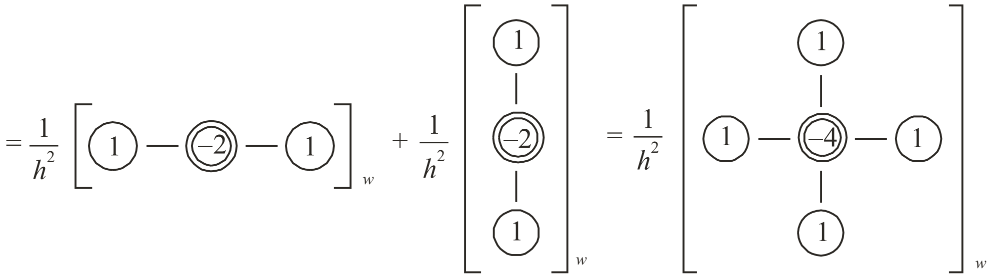
$$\frac{\partial^3 w}{\partial y^3} = \frac{1}{2h^3}\begin{bmatrix} -1 \\ | \\ 2 \\ | \\ 0 \\ | \\ -2 \\ | \\ 1 \end{bmatrix}_w \quad \text{and} \quad \frac{\partial^4 w}{\partial y^4} = \frac{1}{h^4}\begin{bmatrix} 1 \\ | \\ -4 \\ | \\ 6 \\ | \\ -4 \\ | \\ 1 \end{bmatrix}_w$$

11.3 FINITE DIFFERENCE FORM FOR PLATE EQUATION

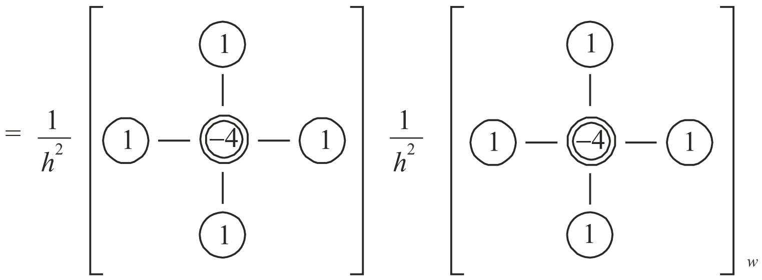
We know,

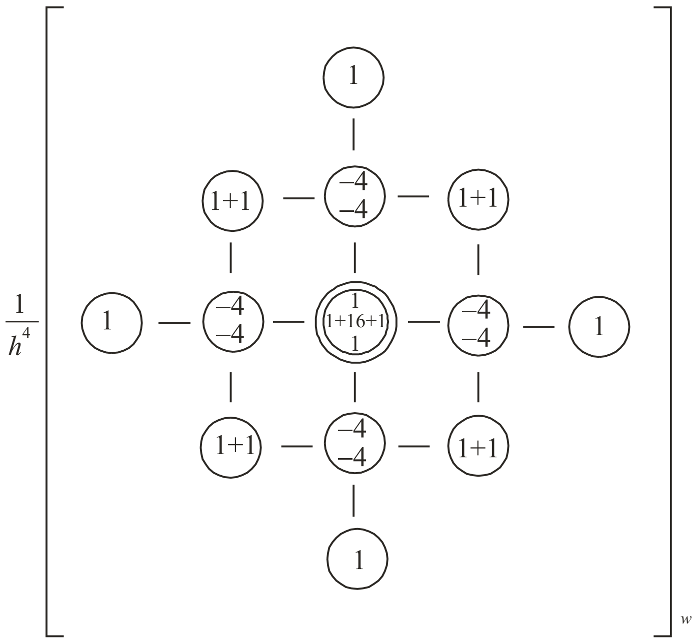
$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{h^2}\begin{bmatrix} 1 & — & -2 & — & 1 \end{bmatrix} \quad \text{and} \quad \frac{\partial^2 w}{\partial y^2} = \frac{1}{h^2}\begin{bmatrix} 1 \\ | \\ -2 \\ | \\ 1 \end{bmatrix}_w$$

$$\therefore \qquad \nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}$$



$$\therefore \qquad \nabla^4 w = \nabla^2 \left(\nabla^2 w\right)$$





$$= \frac{1}{h^4}\left[\begin{array}{ccccc} & & 1 & & \\ & 2 & -8 & 2 & \\ 1 & -8 & 20 & -8 & 1 \\ & 2 & -8 & 2 & \\ & & 1 & & \end{array}\right]_w$$

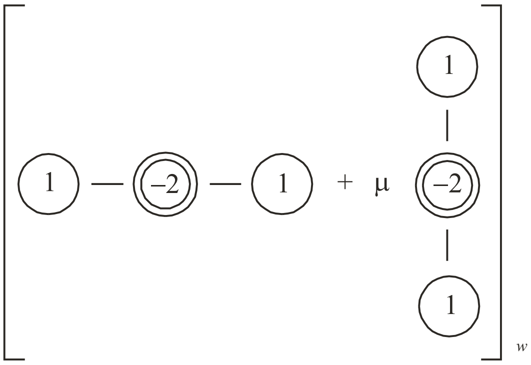
Hence, the plate equation is,

$$\left[\begin{array}{ccccc} & & 1 & & \\ & 2 & -8 & 2 & \\ 1 & -8 & 20 & -8 & 1 \\ & 2 & -8 & 2 & \\ & & 1 & & \end{array}\right]_w = \frac{qh^4}{D} \qquad \text{...eqn. (11.5)}$$

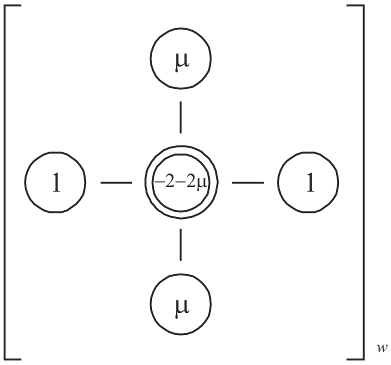
11.4 EXPRESSIONS FOR STRESS RESULTANTS

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2}\right)$$

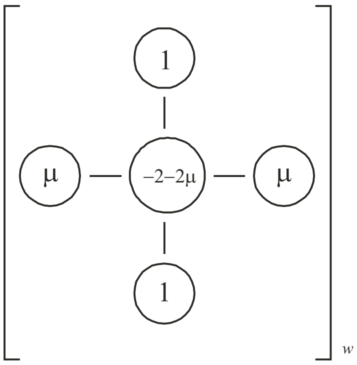
$$= \frac{-D}{h^2}$$



$$= \frac{-D}{h^2}$$



Similarly, $M_y = \frac{-D}{h^2}$



$$M_{xy} = -D(1-\mu)\frac{\partial^2 w}{\partial x \partial y} = -D(1-\mu)\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right)$$

$$= -D\,(1-\mu)\frac{1}{2h}\begin{bmatrix} -1 & - & 0 & - & 1 \end{bmatrix}\frac{1}{2h}\begin{bmatrix} -1 \\ | \\ 0 \\ | \\ 1 \end{bmatrix}_w$$

$$= \frac{-D(1-\mu)}{4h^2}\begin{bmatrix} 1 & - & 0 & - & -1 \\ | & & | & & | \\ 0 & - & 0 & - & 0 \\ | & & | & & | \\ -1 & - & 0 & - & 1 \end{bmatrix}_w$$

It may be easily seen that,

$$Q_x = -D\frac{\partial}{\partial x}\left(\nabla^2 w\right)$$

$$= \frac{-D}{2h^3}\begin{bmatrix} & & -1 & - & 0 & - & 1 & & \\ & & | & & | & & | & & \\ -1 & - & 4 & - & 0 & - & -4 & - & 1 \\ & & | & & | & & | & & \\ & & -1 & - & 0 & - & 1 & & \end{bmatrix}_w$$

$$V_x = Q_x + \frac{\partial}{\partial y}\left(M_{xy}\right)$$

$$= \frac{-D}{2h^3}\left[\begin{array}{ccccc} & -2+\mu & 0 & 2-\mu & \\ -1 & 6-2\mu & \boxed{0} & -6+2\mu & 1 \\ & -2+\mu & 0 & 2-\mu & \end{array}\right]_w$$

By rotating Q_x and V_x patterns by 90°, we get finite difference patterns for Q_y and V_y.

11.5 APPLYING BOUNDARY CONDITIONS

While forming the plate equations at points close to boundary, many points fall outside the plate. The deflections of such imaginary points should be replaced by the deflections of the points on the plate. For this boundary conditions are to be used. In this article, the expressions for the deflections of imaginary points in terms of real points are derived for the following boundaries:

(*a*) Simply supported edge
(*b*) Fixed edge
(*c*) Free edge

(a) Simply supported edge:

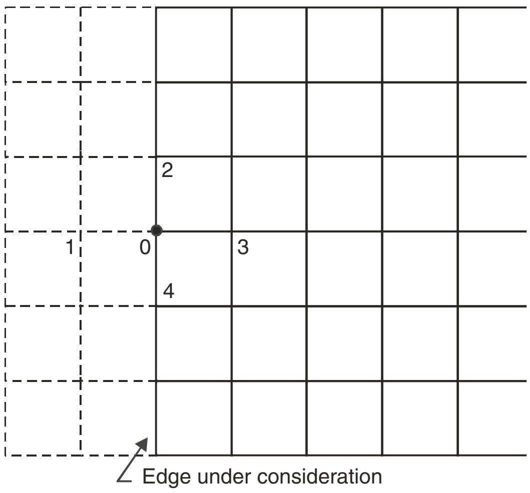


Fig. 11.3

Consider the point '*O*' shown in Fig. 11.3. Since, it is on a simply supported edge

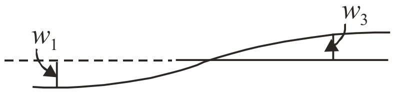
$$M_x|_0 = 0$$

$$\frac{-D}{h^2}\left[\begin{array}{ccccc} & & \textcircled{\mu} & & \\ & & | & & \\ \textcircled{1} & - & \textcircled{\textcircled{-2-2\mu}} & - & \textcircled{1} \\ & & | & & \\ & & \textcircled{\mu} & & \end{array}\right]_w = 0.$$

Since, displacements are zero along simply supported edge, we get

$$w_1 + w_3 = 0$$

i.e. $$w_1 = -w_3$$



i.e. **displacement of imaginary point is equal to negative of that of image point.**

(b) Fixed edge:

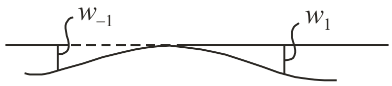
If edge under consideration (Refer Fig. 11.3) is fixed, the boundary conditions are $w = 0$ and $\frac{\partial w}{\partial x} = 0$, along the edge.

$$\left.\frac{\partial w}{\partial x}\right|_0 = 0$$

i.e. $$-\frac{1}{2h}\left[\textcircled{-1} - \textcircled{0} - \textcircled{1}\right]_w = 0$$

$\therefore$ $$-w_1 + w_3 = 0$$

i.e. $$w_1 = w_3$$



i.e. in this case, **displacement of imaginary point is equal to displacement of image point.**

(c) Free edge:

At free edge the boundary conditions are

$M_x = 0$ and $V_x = 0$

$M_x = 0$ gives

$$\left.\begin{array}{ccccc} & & \mu & & \\ & & | & & \\ 1 & — & -2-2\mu & — & 1 \\ & & | & & \\ & & \mu & & \end{array}\right|_w = 0$$

i.e.

$$w_1 = \left.\begin{array}{ccc} -\mu & & \\ | & & \\ 2+2\mu & — & -1 \\ | & & \\ -\mu & & \end{array}\right|_w$$

$V_x = 0$ gives

$$\left.\begin{array}{ccccccccc} & & -2+\mu & — & 0 & — & 2-\mu & & \\ & & | & & | & & | & & \\ -1 & — & 6-2\mu & — & \textcircled{\textcircled{0}} & — & -6+2\mu & — & 1 \\ & & | & & | & & | & & \\ & & -2+\mu & — & 0 & — & 2-\mu & & \end{array}\right|_w = 0$$

$$w_{\text{imaginary}} = \left.\begin{array}{ccccccc} 2-\mu & — & 0 & — & -2+\mu & & \\ | & & | & & | & & \\ -6+2\mu & — & \textcircled{\textcircled{0}} & — & 6-2\mu & — & -1 \\ | & & | & & | & & \\ 2-\mu & — & 0 & — & -2+\mu & & \end{array}\right|_w$$

Note carefully that double circled point is under consideration.

Example 11.1 A fixed plate of size $4h \times 4h$ is subjected to uniformly distributed load q_0 over its entire surface. Taking grid size as $h \times h$ determine.

(*i*) Deflection at the centre of the plate

(*ii*) Moment at the centre of the plate take $\mu = 0.3$

Solution: Making use of symmetry, there are only three unknown displacements w_1, w_2 and w_3 as shown in Fig. 11.4. Displacements are zero along the support.

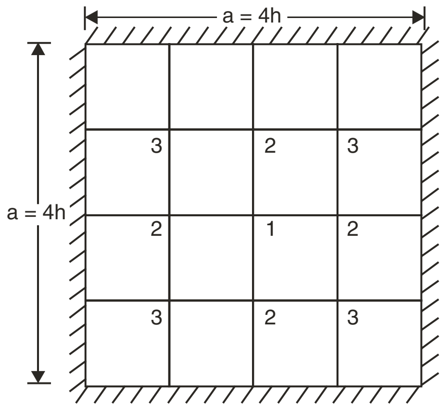


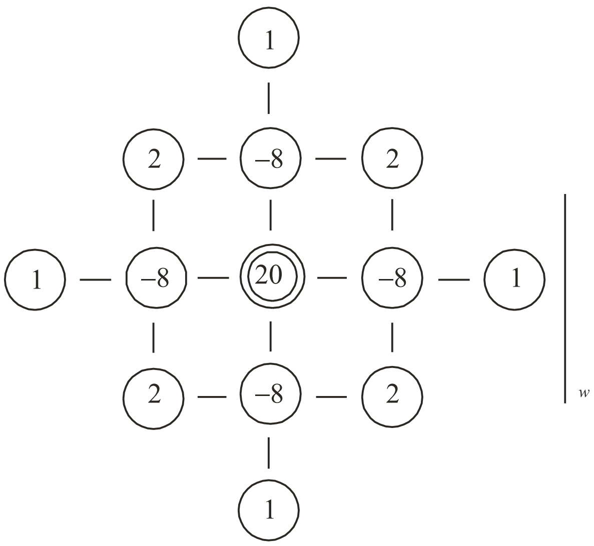
Fig. 11.4 Example 11.1

Since, slope is zero at fixed edges, we get

$$\left[(-1) - (\!(0)\!) - (1) \right]_w = 0$$

$\therefore$ $$w_{-1} = w_1$$

i.e. displacement of imaginary point is equal to displacement of image point. Keeping these points in mind, plate equations can be written for the points 1, 2 and 3. The plate equation is



$$= \frac{qh^4}{D}$$

Hence, the plate equation at point 1 is

$$20w_1 - 4 \times 8w_2 + 4 \times 2w_3 = \frac{q_0 h^4}{D}$$

i.e. $$20w_1 - 32w_2 + 8w_3 = \frac{q_0 h^4}{D} \qquad \ldots(1)$$

The plate equation at point 2 is,

$$20w_2 - 8w_3 - 8w_3 - 8w_1 + 2w_2 + 2w_2 + w_2 + w_2 = \frac{qh^4}{D}$$

i.e. $$-8w_1 + 26w_2 - 16w_3 = \frac{qh^4}{D} \qquad \ldots(2)$$

The plate equation for point 3 is,

$$20w_3 - 8w_2 - 8w_2 + 2w_1 + w_3 + w_3 + w_3 + w_3 = \frac{q_0 h^4}{D}$$

i.e. $$2w_1 - 16w_2 + 24w_3 = \frac{q_0 h^4}{D} \qquad \ldots(3)$$

Solving above three simultaneous equations, we get

$$w_1 = 0.4607 \frac{q_0 h^4}{D}$$

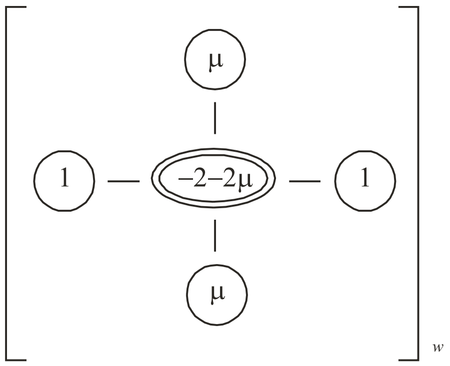
$$w_2 = 0.3090 \frac{q_0 h^4}{D}$$

and $$w_3 = 0.2093 \frac{q_0 h^4}{D}$$

Thus, deflection at centre $= 0.4607 \dfrac{q_0 h^4}{D} = 0.0018 \dfrac{q_0 a^4}{D}$

Moment at centre

$$= \frac{-D}{h^2}$$



$$= -\frac{D}{h^2}\left[(-2-2\mu)w_1 + \mu w_2 + w_2 + \mu w_2 + w_2\right]$$

$$= -\frac{D}{h^2}[-2.6 \times 0.4607 + 2.6 \times 0.3090]\frac{q_0 h^4}{D}$$

$$= 0.3944\ q_0 h^2 = 0.02465\ q_0 a^4$$

Example 11.2 Analyse a simply supported plate of size $4h \times 4h$ which is subjected to *udl* q_0. Take grid size $h \times h$ and determine.

(*a*) Central deflection

(*b*) Moment at the centre if $\mu = 0.3$.

Solution. Due to symmetry there are only three unknown grid point displacements as shown in Fig. 11.5. It is to be noted that

1. Displacements of points on boundary = 0
2. Moment about boundary = 0.

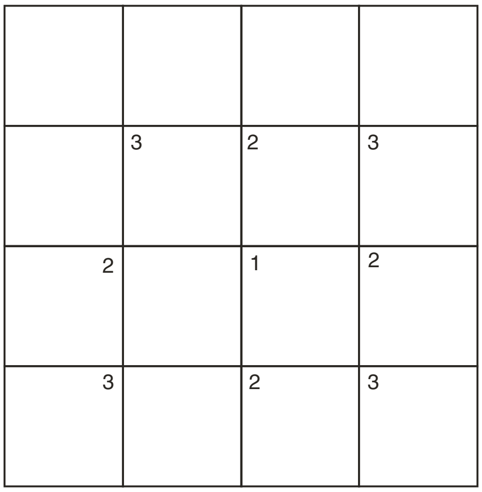
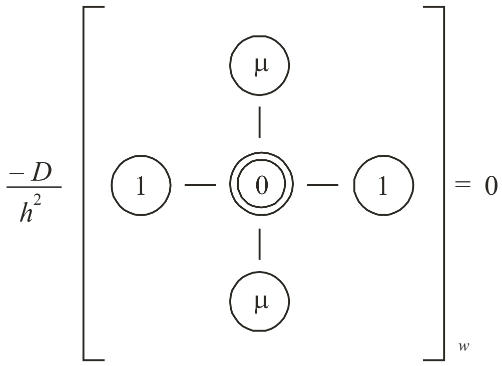


Fig. 11.5 Example 11.2

i.e.



Since, displacements are zero about boundary line this condition gives displacement of an imaginary point is equal to negative of displacement of image point.

Keeping these two points in mind, plate equation is written for the three grid points.

For first point, $20w_1 - 4 \times 8w_2 + 4 \times 2w_1 = \dfrac{q_0h^4}{D}$ ...(1)

For second point, $20w_2 - 8w_3 - 8w_1 - 8w_3 + 2w_2 + 2w_2 + (-w_2) + w_2 = \dfrac{q_0h^4}{D}$

i.e. $$-8w_1 + 24w_2 - 16w_3 = \frac{q_0h^4}{D} \quad \text{...(2)}$$

For point three,

$$20w_3 - 8w_2 - 8w_2 + 2w_1 - w_3 - w_3 + w_3 + w_3 = \frac{q_0h^4}{D}$$

i.e. $$2w_1 - 16w_2 + 20w_3 = \frac{q_0h^4}{D} \quad \text{...(3)}$$

Solving simultaneous equations 1, 2 and 3, we get

$$w_1 = 1.0313\ \frac{q_0h^4}{D}$$

$$w_2 = 0.75\ \frac{q_0h^4}{D}$$

and $$w_3 = 0.5469\ \frac{q_0h^4}{D}$$

Thus, deflection at centre $= w_1 = 1.0313\ \dfrac{q_0h^4}{D}$

$$= 0.0645\frac{q_0a^4}{D}$$

Moment at centre, if $\mu = 0.3$ is given by

$$= \frac{-D}{h^2}\left[\begin{array}{ccc} & \mu & \\ 1 & -2-2\mu & 1 \\ & \mu & \end{array}\right]_w$$

$$= -\frac{D}{h^2}\left[(-2-2\mu)w_1 + \mu w_2 + w_2 + \mu w_2 + w_2\right]$$

$$= -\frac{D}{h^2}\left[-2.6w_1 + 2.6w_2\right]$$

$$= [-2.6 \times 1.0313 + 2.6 \times 0.75]q_0h^2$$

$$= 0.7314q_0h^2 = 0.0457\,q_0a^4$$

Example 11.3 Formulate finite difference equations for the plate shown in Fig. 11.6. Take grid size $h \times h$.

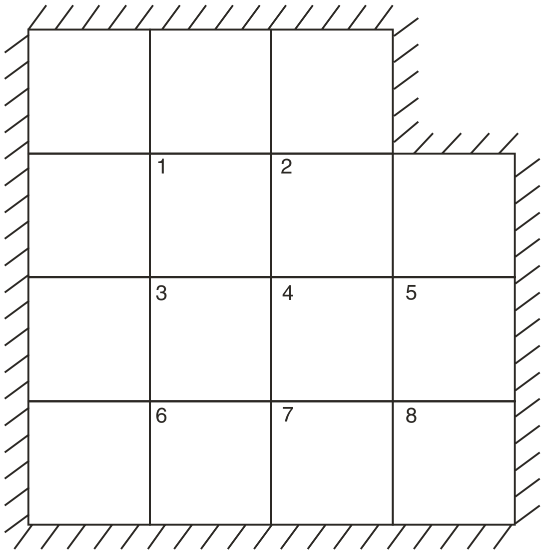


Fig. 11.6 Example 11.3

Solution. In this problem, deflections of all interior grid points are different. Hence, eight separate equations are to be written. Noting that all edges are fixed

$w = 0$ at points on edges

and deflection of imaginary point = Deflection of image point.

Keeping these points in mind the eight equations are written.

For point 1,

$$20w_1 - 8w_2 - 8w_3 + 2w_4 + w_1 + w_1 + w_6 = \frac{q_0h^4}{D}$$

$$22w_1 - 8w_2 - 8w_3 + w_6 = \frac{q_0h^4}{D} \qquad \ldots(1)$$

For point 2,

$$20w_2 - 8w_4 - 8w_1 + 2w_5 + 2w_3 + w_2 + w_7 = \frac{q_0h^4}{D}$$

i.e. $$-8w_1 + 21w_2 + 2w_3 - 8w_4 + 2w_5 + w_7 = \frac{q_0h^4}{D} \quad \ldots(2)$$

For point 3,

$$20w_3 - 8w_1 - 8w_4 - 8w_6 + 2w_2 + 2w_7 + 2w_5 + w_3 = \frac{q_0h^4}{D}$$

i.e. $$-8w_1 + 2w_2 + 21w_3 - 8w_4 + 2w_5 - 8w_6 + 2w_7 = \frac{q_0h^4}{D} \quad \ldots(3)$$

For point 4,

$$20w_4 - 8w_2 - 8w_5 - 8w_7 - 8w_3 + 2w_1 + 2w_8 + 2w_6 = \frac{q_0h^4}{D}$$

i.e. $$2w_1 - 8w_2 - 8w_3 + 20w_4 - 8w_5 + 2w_6 - 8w_7 + 2w_8 = \frac{q_0h^4}{D} \quad \ldots(4)$$

For point 5,

$$20w_5 - 8w_8 - 8w_4 + 2w_2 + 2w_7 + w_5 + w_3 = \frac{q_0h^4}{D}$$

i.e. $$2w_2 + w_3 - 8w_4 + 20w_5 + 2w_7 - 8w_8 = \frac{q_0h^4}{D} \quad \ldots(5)$$

For point 6,

$$20w_6 - 8w_3 - 8w_7 + 2w_4 + w_1 + w_8 + w_6 + w_6 = \frac{q_0h^4}{D}$$

i.e. $$w_1 - 8w_3 + 2w_4 + 22w_6 - 8w_7 + w_8 = \frac{q_0h^4}{D} \quad \ldots(6)$$

For point 7,

$$20w_7 - 8w_4 - 8w_8 - 8w_6 + 2w_3 + 2w_5 + w_2 + w_7 = \frac{q_0h^4}{D}$$

$$w_2 + 2w_3 - 8w_4 + 2w_5 - 8w_6 + 21w_7 - 8w_8 = \frac{q_0h^4}{D} \quad ...(7)$$

For point 8,

$$20w_8 - 8w_5 - 8w_7 + 2w_4 + w_8 + w_8 + w_6 = \frac{q_0h^4}{D}$$

$$2w_4 - 8w_5 + w_6 - 8w_7 + 22w_8 = \frac{q_0h^4}{D} \quad \ldots(8)$$

11.6 MODIFICATION OF FINITE DIFFERENCE PLATE EQUATION TO APPLY IT ON A FREE EDGE

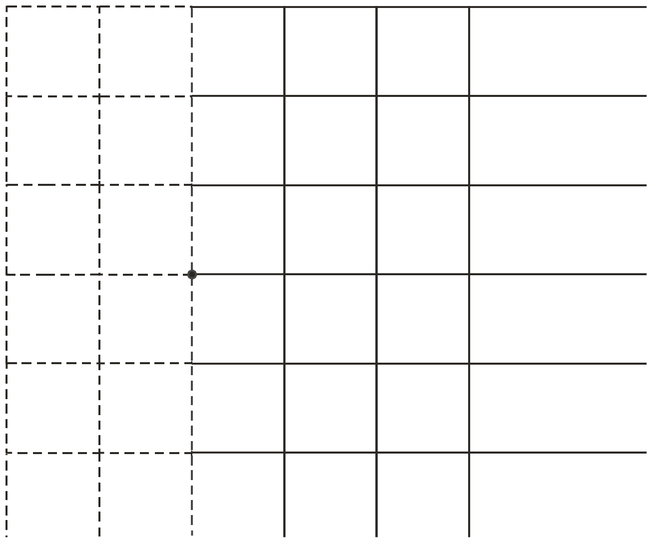
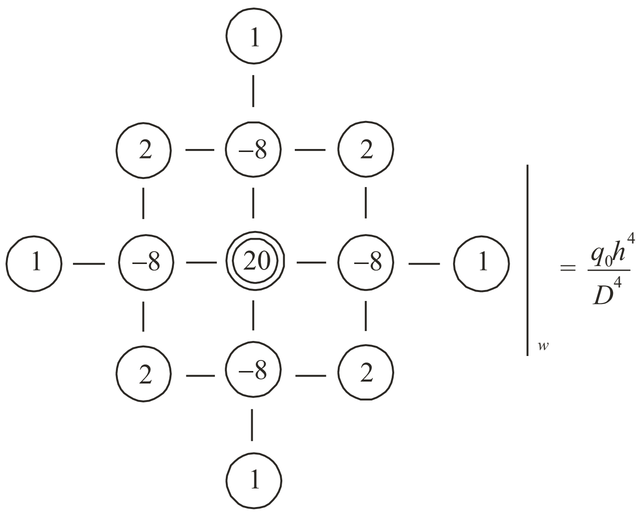


Fig. 11.7 Modifying plate equation so as to apply at point ‘O’

The plate equation for any point is

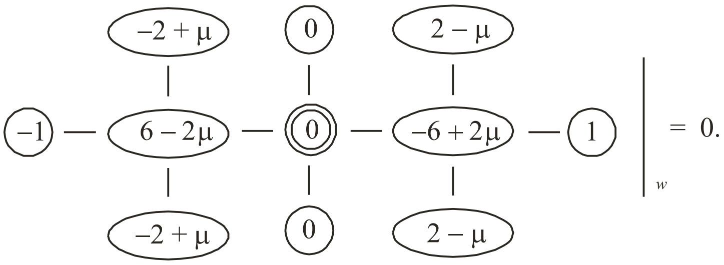


When this equation is to be applied at point, four points fall away from the plate, one at distance $2h$ and three at distance h from free edge of the plate. The plate equation can be modified making use of the boundary conditions that

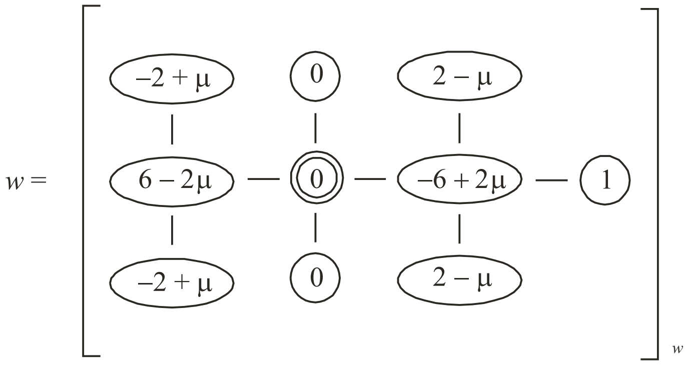
$$V_x = 0$$

and $M_x = 0$ at free edges.

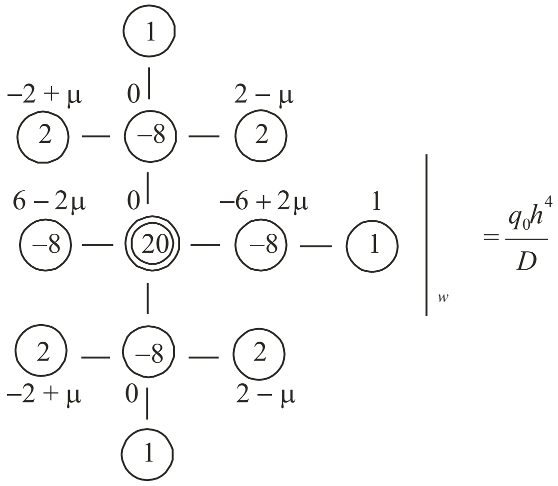
Using the boundary condition $V_x = 0$



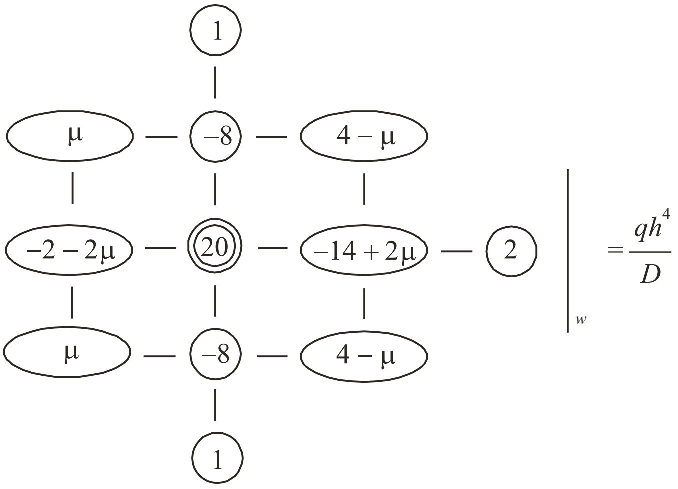
i.e.



Substituting it in plate equation, we get

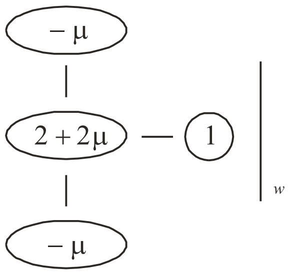


i.e.

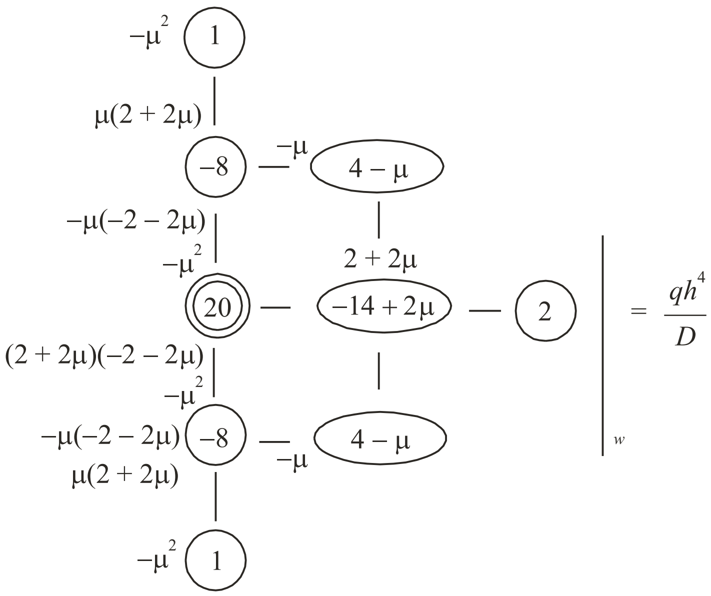


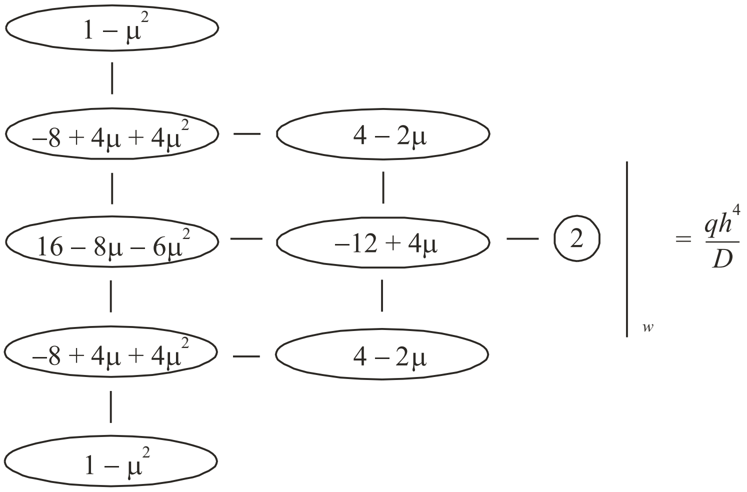
From boundary condition $M_x = 0$, we get

Displacement of imaginary point =



Applying this condition to the three points which are at distance '*h*' from free edge on fictitious plate we get





QUESTIONS

1. Derive the plate equation in finite difference form.
2. Modify the plate equation so as to apply it on a boundary with free edge.
3. Write plate equations in finite difference form for the plate shown in Fig. 11.8.
   [**Hint:** Note due to symmetry $w_1 = w_7$, $w_2 = w_8$, $w_3 = w_9$]

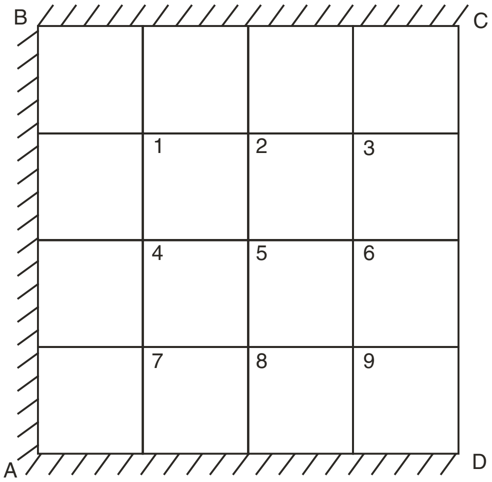


Fig. 11.8 Question No. 3

4. In the plate shown in Fig. 11.8, if the edge is simply supported, write plate equations.
   [**Hint:** Note due to anti-symmetry $w_1 = w_9$, $w_2 = w_6$, $w_4 = w_8$]

A Folded Plate Roofs

Folded plate roof is a structure which is composed of a number of plates monolithic along their common longitudinal edges. It may be looked as a plate folded at several longitudinal lines. The roof unit is supported at ends by thin but deep frames. Such support may be treated as simple support (Fig. 12.1). The end frames in turn are supported by columns. Figure 12.2 shows commonly used folded plate roofs.

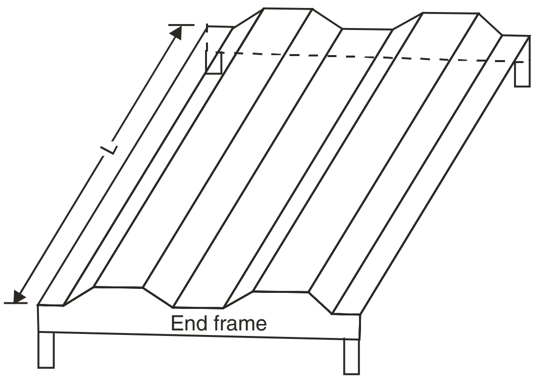


Fig. 12.1 A typical folded plate roof

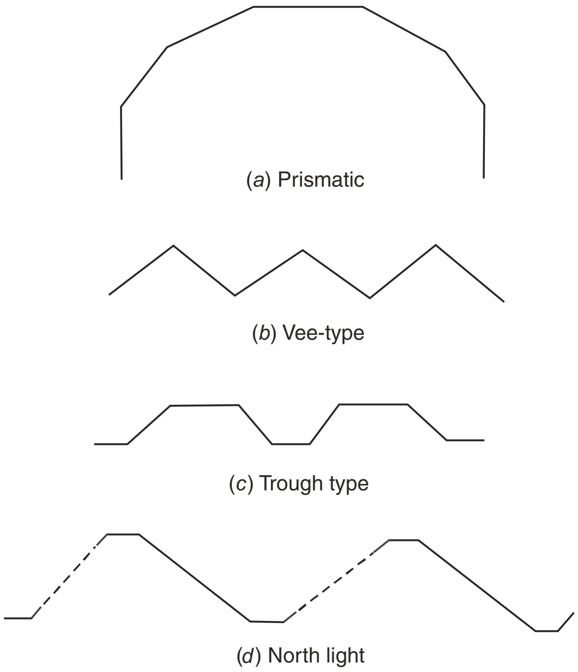


Fig. 12.2 Types of folded plate roofs

Folded plate roofs are known as **hipped plates, prismatic shells** and **faltwerke.** By giving folds to the plate, bending moment is reduced in the plate and considerable load is transferred as membrane compressions. Hence, they can be used economically to cover a column free span of 20 to 25m. However, folded plates are not as efficient as shells in transferring loads by membrane compression.

12.1 ADVANTAGES AND DISADVANTAGES OF FOLDED PLATES OVER SHELLS

The advantages of folded plates over shells are:

1. Form work required is relatively simple.
2. Movable form can be employed.
3. Design involves simpler calculations.

The disadvantages of folded plates over shells are:

1. Folded plates consume a little more material than shells.
2. Form work can be removed after 7 days of concreting while in case of shells it may be removed after 3 days only.

12.2 HISTORICAL NOTE

Ehler of Germany published first paper on folded plate bunkers in 1930. He assumed the joints at folds as hinged. In 1934, Gruber put forward a rigorous theory which took into account the rigidity of the joints and the relative displacement between them. In the course of next few years, many European engineers made contributions to the subject—noteworthy among them being by Cramer Ohlig and Girkman. The methods developed were based on theory of elasticity. Hence, they involve differential and algebraic equations.

After world war II, American took interest in folded plate analysis. In 1947, Winter and Pei published stress distribution procedure. It ignored relative displacements of the joints. In 1954, Gaafer published a simple method for analysis, which took into account joint displacements also. Later on Simpsons improved it and now that method is known as Simpson's method. Girkmans method, improved by Whitney is known as Whitney's method.

12.3 ASSUMPTIONS

The following assumptions are made in the analysis of folded plates:

1. It consists of rectangular plates each being of uniform thickness.
2. The structure is monolithic and the joints are rigid.
3. The material is elastic, homogeneous and isotropic.
4. The length of each plate is more than twice its width.
5. In all plates, plane section remains plane even after deformation.

12.4 BEHAVIOUR OF FOLDED PLATE ROOFS

Load transfer in folded plate roofs consists of slab action and plate action.

12.4.1 Slab Action

Each slab may be assumed to bend as simply supported slab between the adjacent folds. It results into the reactions at joints (folds). But actually the slab is continuous over folds. Hence, there are end moments in transverse direction. Taking joint moments as unknowns, joint-reactions may be found in terms of unknown moments. Thus, total reactions at joints if they are supported externally are the algebraic sum of reactions due to load plus those due to transverse moments. Figure 12.3 shows slab action of a typical folded plate roof.

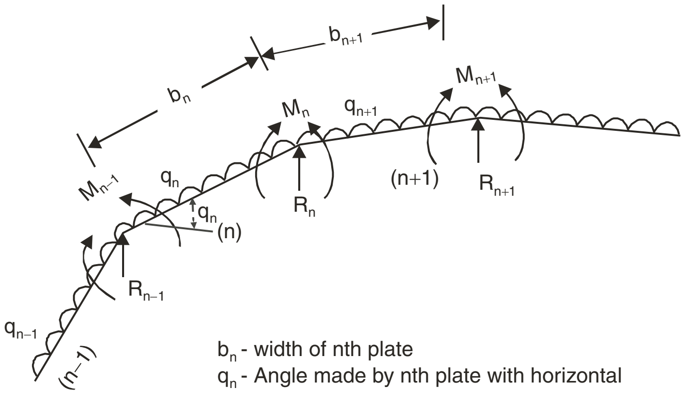


Fig. 12.3

12.4.2 Plate Action

Actually there is no external supports at folds. Hence, the joint reactions found due to slab action are reversed and applied as joint force. This joint loads get distributed as plate load as shown in Figure 12.4.

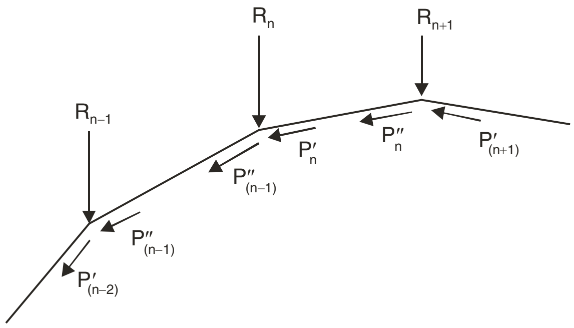


Fig. 12.4

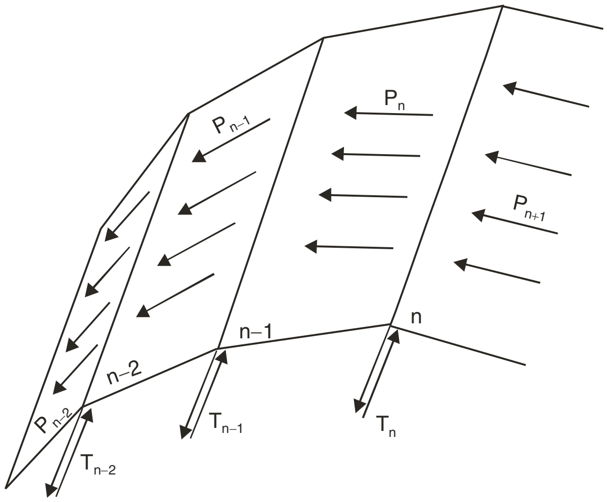


Fig. 12.5

Assuming each plate is simply supported on end frames, bending moment in the plate due to plate loads can be found. This plate bending results into discontinuity at joints. Since, the joints are rigid, this discontinuity is not possible. This has happened because longitudinal shears developed at joints are ignored. Let the longitudinal shears developed be as shown in Fig. 12.5.

Taking longitudinal shears also as unknowns, equations are developed to find stresses at edges of each plate. Using compatibility conditions, unknown transverse moments and longitudinal shears are determined.

There are two different methods for the analysis of plate. They are known as Whitney's method and Simpson's method. In Whitney's method simultaneous equations are solved to get transverse moments and longitudinal shears whereas in Simpson's method effect of transverse moments is taken care by moment distribution procedure while the effect of longitudinal shears is taken care by a stress distribution procedure. These methods are explained in this chapter.

12.5 WHITNEY'S METHOD

At each joint, there are two unknowns—a transverse moment and a longitudinal shear. Hence, if there are N number of joints (*i.e.* $N + 1$ number of plates), there are $2N$ number of unknowns. These unknowns at joints $n - 1$, n and $n + 1$ are shown in Fig. 12.6.

The transverse moments at joint 1 and joint N may be taken as those due to cantilever action of 1st and $n + 1$th plate. Hence, two of the $2N$ unknowns are found. Thus, number of unknown reduce to $2N–2$. In Whitney's method, N-number of equations are obtained by equating stresses in adjoining plates at common edges.

Another compatibility condition to be satisfied is at any joint the angle between the two adjacent plates should not change. This condition may be applied from joint 2 to joint $N–1$ to get $N–2$ equations.

Thus, the compatibility of stresses and transverse rotations gives $2N–2$ equations. Hence, $2N–2$ unknowns, namely, $M_2, M_3, ..., M_{N-1}$ and $T_1, T_2, ..., T_N$ can be determined.

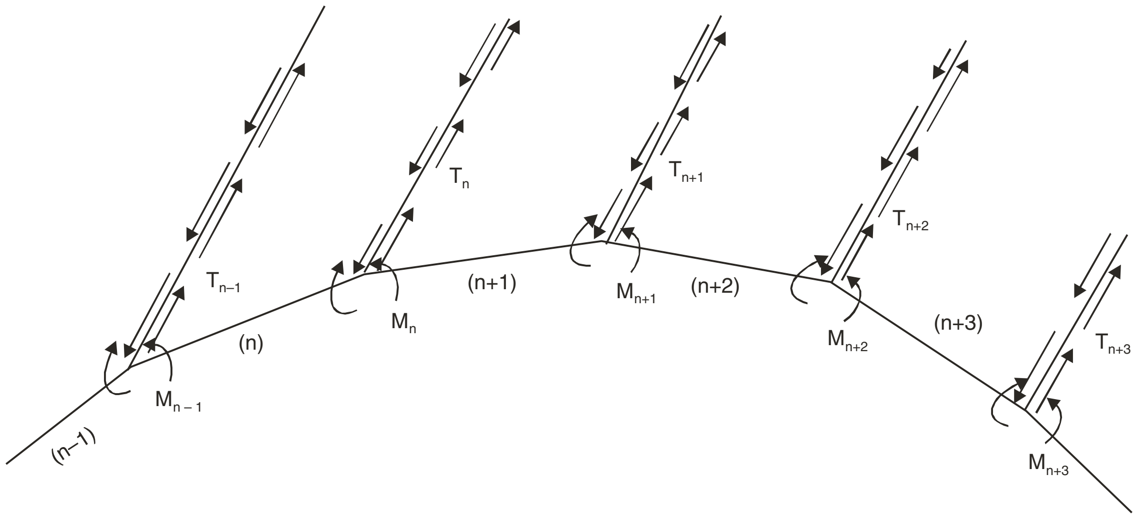


Fig. 12.6 Unknowns in Whitney's method

The step by step procedure of analysis of folded plates by Whitney's method is given below:

Step. 1: Express the loading in the Fourier form:

$$q(x) = \sum \frac{4}{m\pi} q_0 \sin \frac{m\pi x}{a}$$

If only first term is considered,

$$q(x) = \frac{4q_0}{\pi} \sin \frac{\pi x}{L} = q_0 \sin \frac{\pi x}{L} \qquad \text{...eqn. 12.1}$$

Step 2: Assuming plates are simply supported along the folds, find reactions (Refer Fig. 12.7)

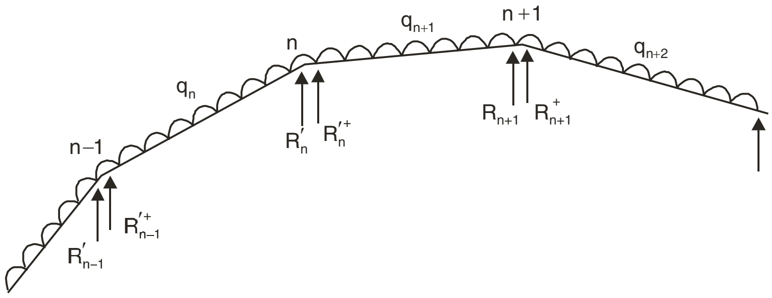


Fig. 12.7 Load and reactions at mid span due to loads for slab action

$$q_n = \frac{q_n b_n + q_{n+1} b_{n+1}}{2} \sin \frac{\pi x}{L} \qquad \text{...eqn. 12.2}$$

Step 3: To be compatible with sinusoidal loading transverse moments should vary in the sinusoidal form. Then the reaction at joint n due to transverse moments (Refer Fig. 12.8) are

$$R''_n = \left[\frac{M_n - M_{n-1}}{b_n \cos\phi_n} + \frac{M_n - M_{n+1}}{b_{n+1} \cos\phi_{n+1}} \right] \sin \frac{\pi x}{L} \qquad \text{...eqn. 12.3}$$

Step 4: Calculate total reaction at the imaginary support along the folds.

$$R_n = R'_n + R''_n \qquad \text{...eqn. 12.4}$$

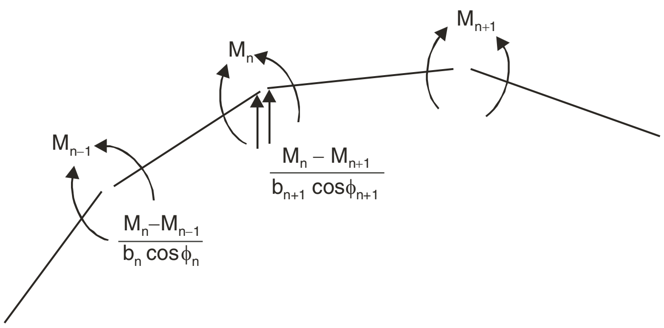
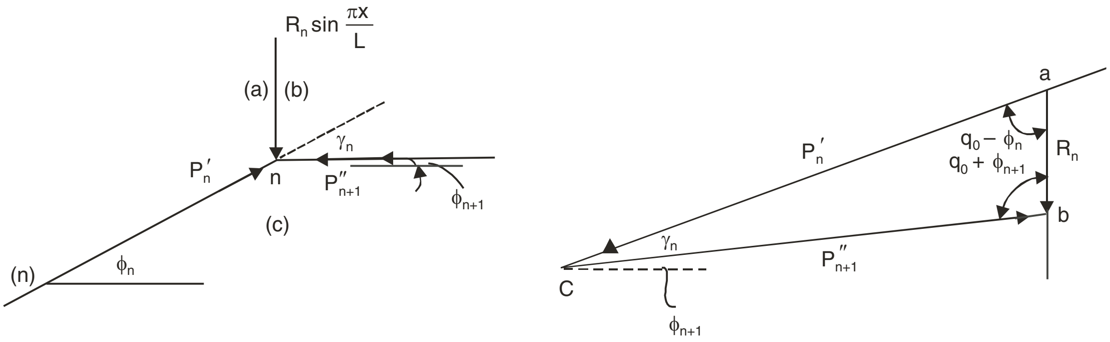


Fig. 12.8 Reactions due to transverse moment at mid span at joint *n*

Step 5: Actually there is no external support at folds. Hence, the joint reaction R_n acts as joint load in the downward direction. These joint loads are transferred to the plates meeting at joints causing each plate to bend as a beam spanning between the end traverses.



(*a*) R_n and its components at joint n (*b*) Vector diagram

Fig. 12.9 Finding components of R_n

Referring to Fig. 12.9, if *ab* represents the joint load R_n, its components in the direction of *n*th and *n*th plate are obtained by traversing from *a* to *b* in the directions of *n*th plate and (*n*+1)th plate, *ac* represents the component in *n*th plate while *cb* represents the component in (*n*+1)th plate. From triangle of forces *abc,* we get

$$P_n' = ac = \frac{R_n \sin\dfrac{\pi x}{L}}{\sin\gamma_n}\sin(90+\phi_{n+1}) = \frac{R_n}{\sin\gamma_n}\cos\phi_{n+1}\cdot\sin\frac{\pi x}{L}$$

$$P_{n+1}'' = cb = \frac{R_n \sin\dfrac{\pi x}{L}}{\sin\gamma_n}\sin(90-\phi_n) = \frac{R_n}{\sin\gamma_n}\cos\phi_n\cdot\sin\frac{\pi x}{L}$$

Similarly, if we resolve the joint load R_{n-1} at joint $n-1$, the load P_n'' is given by

$$P_n'' = \frac{R_{n-1}\sin\dfrac{\pi x}{L}}{\sin r_{n-1}}\cos\phi_{n-1}$$

∴ Total force in *n*th plate from *n* to $n-1$ direction is (Ref. Fig. 12.5)

$$P_n = P_n' - P_n'' = \left[\frac{R_n\cos\phi_{n+1}}{\sin\gamma_n} - \frac{R_{n-1}}{\sin\gamma_{n-1}}\cdot\cos\phi_{n-1}\right]\sin\frac{\pi x}{L} \qquad \text{...eqn. 12.5}$$

where γ_n is the angle of deviation of $(n+1)$th plate from the direction of *n*th plate. From equation 12.5, plate loads in all plates may be found.

Step 6: Since, the longitudinal variation of plate loads are in sinusoidal form, the plate moment also should be in the sinusoidal form. Thus,

$$M_{pn}' = P_n\frac{L^2}{\pi^2}\sin\frac{\pi x}{L} \qquad \text{...eqn. 12.6}$$

Step 7: The moment due to the longitudinal shears T_{n-1} and T_n at the centre of the plate (Refer Fig. 12.10) is

$$M_{pn}'' = (T_{n-1}+T_n)\frac{b_n}{2}\cdot\sin\frac{\pi x}{L} \qquad \text{...eqn. 12.7}$$

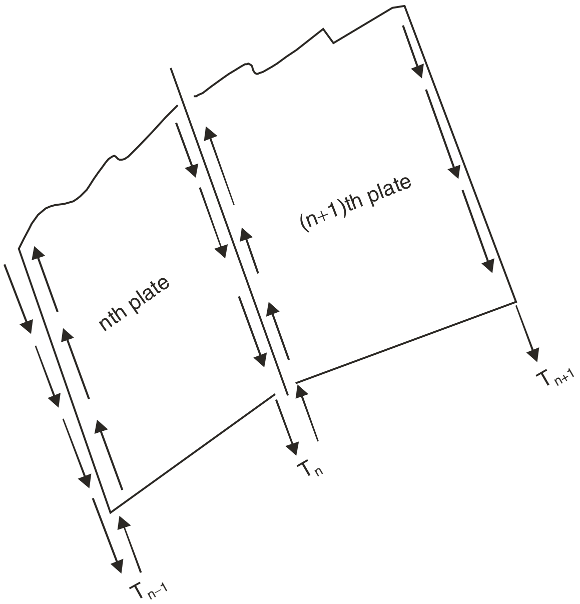


Fig. 12.10

It may be noted that M_{pn} to be in $\sin\frac{\pi x}{L}$ form, T_n also should vary in sinusoidal form.

Step 8: Total plate moment is

$$M_{pn} = \left[\frac{P_n L^2}{\pi^2} + (T_{n-1} + T_n)\frac{b_n}{2}\right]\sin\frac{\pi x}{L} \qquad \text{...eqn. 12.8}$$

Step 9: Writing tensile stress as positive, the fibre stress in nth plate at common edge n is,

$$= \frac{6}{b_n h_n^2}\left[-\frac{P_n L^2}{\pi^2} + (T_{n-1} + T_n)\frac{b_n}{2}\right]\sin\frac{\pi x}{L} + \frac{T_n - T_{n-1}}{b_n h_n}\sin\frac{\pi x}{L} \qquad \text{...eqn. 12.9}$$

where first part is due to flexure while second part is due to direct force $(T_{n+1} - T_n)$.

Similarly, the tensile stress at the same fibre calculated from $(n + 1)$th plate is

$$= \frac{6}{b_{n+1} h_{n+1}^2}\left[P_{n+1}\frac{L^2}{\pi^2} - (T_n + T_{n+1})\frac{b_{n+1}}{2}\right]\sin\frac{\pi x}{L} + \frac{T_{n+1} - T_n}{b_{n+1} h_{n+1}}\sin\frac{\pi x}{L} \qquad \text{...eqn. 12.10}$$

To maintain stress compatibility, equation 12.9 should be equal to equation 12.10. Hence,

$$\frac{6}{b_n h_n^2}\left[-P_n\frac{L^2}{\pi^2} + (T_{n-1} + T_n)\frac{b_n}{2}\right] + \frac{T_n - T_{n-1}}{b_n h_n}$$

$$= \frac{6}{b_{n+1} h_{n+1}^2}\left[P_{n+1}\frac{L^2}{\pi^2} - (T_n + T_{n+1})\frac{b_{n+1}}{2}\right] + \frac{T_{n+1} - T_n}{b_{n+1} h_{n+1}} \qquad \text{...eqn. 12.11}$$

Applying the above compatibility condition from joint 1 to joint *N*, *N* equations can be assembled. Equation 12.11 is known as **Equation of Three shears**.

Step 10: Rotation of *n*th plate and (*n*+1)th plate due to various causes are,

(*a*) Due to slab action, for applied loads

$$\alpha_{n,n-1} = -\frac{1}{24}\frac{q_n b_n^3}{EI_n'}\cos\phi_n \sin\frac{\pi x}{L}$$

$$\alpha_{n,n+1} = \frac{1}{24}\frac{q_{n+1} b_{n+1}^3}{EI_{n+1}'}\cos\phi_{n+1} \sin\frac{\pi x}{L}$$

where $I_n' = \frac{1}{12}h_n^3$, moment of inertia per unit length of plate.

(*b*) Due to slab action, for transverse moments

$$\beta_{n,\,n-1} = \frac{b_n}{6EI_n'}\left[2M_n + M_{n-1}\right]\sin\frac{\pi x}{L}$$

$$\beta_{n,\,n+1} = -\frac{b_{n+1}}{6EI_{n+1}'}\left[2M_n + M_{n+1}\right]\sin\frac{\pi x}{L}$$

(*c*) Slope due to plate deflection:

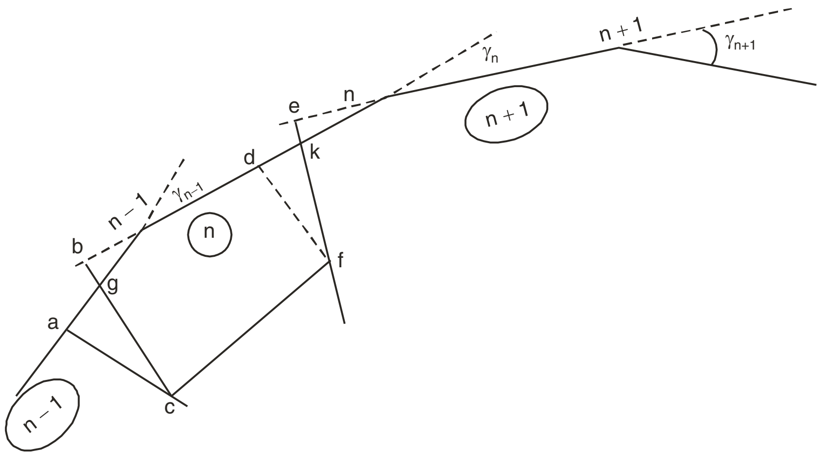


Fig. 12.11

Referring to Fig. 12.11, let Δ be the free deflection of plates. Then,

$$\Delta_{n-1} = n - 1,\ a$$

$$\Delta_n = n - 1,\ b = n,\ d$$

$$\Delta_{n+1} = n,\ e$$

Since, the joints are rigid, common edges cannot be at two different points, in other words, free expansion of plates is not possible. Hence, joint *n*–1 moves to the point *c* and the joint *n* moves to the point *f*, which are located by dropping perpendiculars from *a, b* and *e, d* respectively. Line *cf* shows the position of the plate after deflection.

(i) To find bc:

From triangle $n-1$, b, g, we get

$$n-1, g = \frac{n-1, b}{\cos\gamma_{n-1}} = \frac{\Delta n}{\cos\gamma_{n-1}}$$

$$\therefore \quad a, g = n-1, a — n-1, g$$

$$= \Delta_{n-1} - \frac{\Delta_n}{\cos\gamma_{n-1}}$$

$$\therefore \quad gc = \frac{ag}{\sin\gamma_{n-1}} = \frac{1}{\sin\gamma_{n-1}}\left[\Delta_{n-1} - \frac{\Delta_n}{\cos\gamma_{n-1}}\right]$$

$$\therefore \quad bc = bg + gc$$

$$= \Delta_n \tan\gamma_{n-1} + \frac{1}{\sin\gamma_{n-1}}\left[\Delta_{n-1} - \frac{\Delta_n}{\cos\gamma_{n-1}}\right]$$

(ii) To find df:

$$nk = \frac{en}{\cos\gamma_n} = \frac{\Delta_{n+1}}{\cos\gamma_n}$$

$$\therefore \quad dk = dn - nk$$

$$= \Delta_n - \frac{\Delta_{n+1}}{\cos\gamma_n}$$

$$\therefore \quad df = dk \cot\gamma_n$$

$$= \left[\Delta_n - \frac{\Delta_{n+1}}{\cos\gamma_n}\right]\cot\gamma_n$$

Now, rotation of nth plate is given by

$$\theta_n = \frac{1}{b_n}(df - bc)$$

$$= \frac{1}{b_n}\left[\left(\Delta_n - \frac{\Delta_{n+1}}{\cos\gamma_n}\right)\cot\gamma_n - \Delta_n \tan\gamma_{n-1} - \frac{1}{\sin\gamma_{n-1}}\left(\Delta_{n-1} - \frac{\Delta_n}{\cos\gamma_{n-1}}\right)\right]$$

$$= \frac{1}{b_n}\left[\Delta_n\left(\cot\gamma_n - \tan\gamma_{n-1} + \frac{1}{\sin\gamma_{n-1}\cos\gamma_{n-1}}\right) - \frac{\Delta_{n-1}}{\sin\gamma_{n-1}} - \frac{\Delta_{n+1}}{\cos\gamma_n}\cot\gamma_n\right]$$

$$= \frac{1}{b_n}\left[\Delta_n\left(\cot\gamma_n + \frac{1-\sin^2\gamma_{n-1}}{\sin\gamma_{n-1}\cos\gamma_{n-1}}\right) - \frac{\Delta_{n-1}}{\sin\gamma_{n-1}} - \frac{\Delta_{n+1}}{\sin\gamma_n}\right]$$

$$= \frac{1}{b_n}\left[\Delta_n\left(\cot\gamma_n + \cot\gamma_{n-1}\right) - \frac{\Delta_{n-1}}{\sin\gamma_{n-1}} - \frac{\Delta_{n+1}}{\sin\gamma_n}\right]$$

Similarly,

$$\theta_{n+1} = \frac{1}{b_{n+1}}\left[\Delta_{n+1}\left(\cot\gamma_{n+1} + \cot\gamma_n\right) - \frac{\Delta_n}{\sin\gamma_n} - \frac{\Delta_{n+2}}{\sin\gamma_{n+1}}\right]$$

Since, joints are rigid, the following compatibility condition should be satisfied:

$$\alpha_n + \beta_n + \theta_n = \alpha_{n+1} + \beta_{n+1} + \theta_{n+1} \quad \text{...eqn. 12.12}$$

Equation 12.12 may be applied from joint 2 to joint $N - 1$ to get $N - 2$ equations.

Step 11: Thus, from equations 12.11 and 12.12, we get $2N - 2$ equations in $2N - 2$ unknowns. Solving simultaneous equations the unknowns M_2, M_3, ..., M_{N-1}, T_1, T_2, ..., T_N can be found.

Step 12: After knowing transverse moments, reinforcement required in transverse direction may be found. Spacing may be increased towards end frames, since, these moments vary in the form $\sin\frac{\pi x}{L}$. Knowing longitudinal stresses from equation 12.9, longitudinal reinforcement may be decided. From plate action, end shear may be found and the diagonal shear reinforcement decided. Figure 12.1 shows typical reinforcements in a tough type folded plate.

Note: Many times instead of bending transverse steel up and down, two layers of transverse steel are provided longitudinal steels are located so as to support transverse steel.

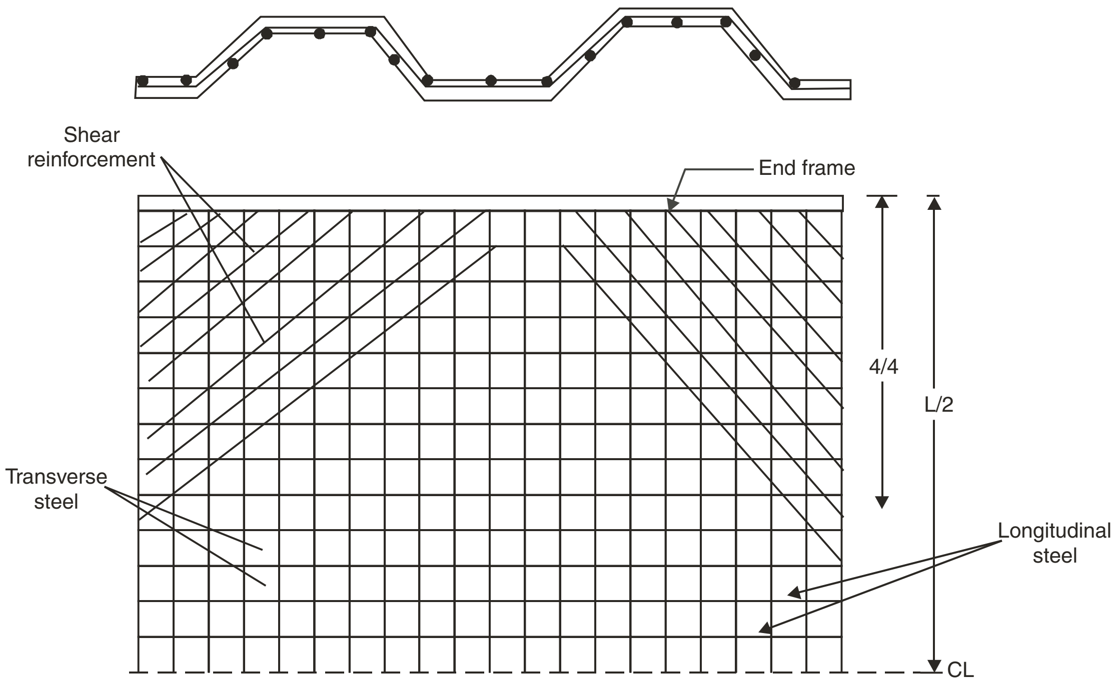


Fig. 12.12 R.C. Details of a folded plate roof

12.6 SIMPSONS METHOD

In Whitney's method, there is need for solving $2N - 2$ simultaneous equations to get transverse moments and longitudinal shears at folds. In Simpson's method, solution of simultaneous equations is avoided by going through moment distribution and stress distribution procedure. Stress distribution procedure is similar to moment distribution. It is used to ensure the continuity of stresses calculated for plate action neglected the effect of longitudinal stresses. This technique is in two parts—first part consists in finding stress continuity when the joints do not deflect from their original positions and the second part allowing joint deflections. The method is outlined below with reference to the folded plate shown in Fig. 12.13.

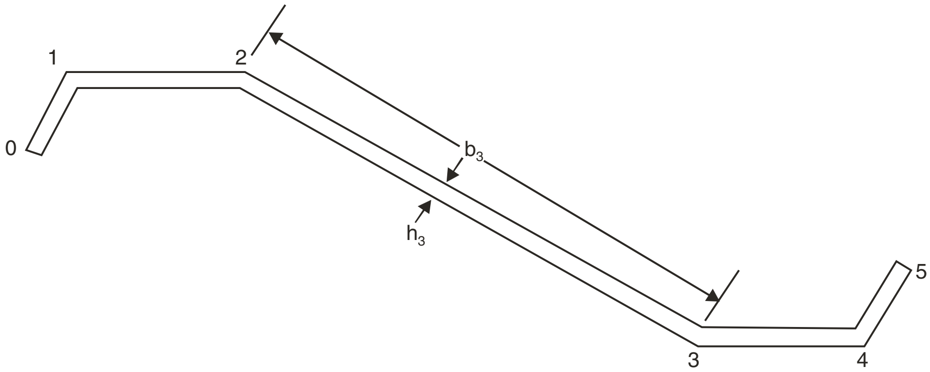


Fig. 12.13

Step 1: Consider a transverse section of unit length at mid span. Assuming that the joints do not deflect arrive at joint moments by moment distribution. Calculate the reactions at the joints and apply forces equal and opposite to these as joint loads. Resolve these joint loads into plate loads. Calculate the bending stresses caused by the plate loads, assuming each plate to be free to bend independently. These stresses may be referred as free edge stresses.

Next establish stress compatibility at the common edges of adjacent plates by stress distribution. The resulting stresses in the plates are those which develop if the joints do not deflect. This solution is referred as no rotation solution. The solution up to this point may be referred as Winter and Pei solution, Winter and Pei are the researchers who stopped the analysis at this stage.

Step 2: The effect of joint displacements is to be accounted by considering the rotations of plates 2, 3 and 4. The first and the last plates are treated as cantilevers.

Let joint 2 deflect by an arbitrary amount Δ_{20} below the level of joint 1 (Fig. 12.14). As a result of it fixed end moment developed at joint 2 is $\dfrac{3EJ_2\Delta_{20}}{b_2^2} = \dfrac{3EJ_2\Psi_{20}}{b_2}$, where J_2 is moment of inertia of the plate $\left(= \dfrac{1}{12}h^3\right)$ per unit length and $\Psi_{20} = \dfrac{\Delta_{20}}{b_2}$. As Δ_{20} is arbitrary, ψ_{20} is an arbitrary rotation of plate 2. Let ψ_{20} be such that the magnitude of the moment introduced is, say 300 units. The arbitrary rotations and the actual rotations of the plate are clearly related by an unknown constant, say K_2, such that $\psi_2 = K_2\psi_{20}$. The arbitrary moment of 300 at joint 2 is next distributed by the moment distribution

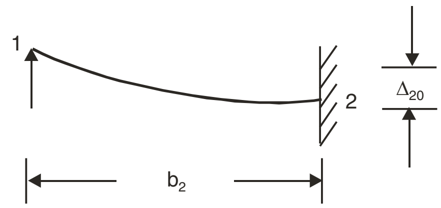


Fig. 12.14

procedure. The resulting joint moments and reactions are found. Forces equal and opposite to the reactions are applied at the joints and resolved along the plates as plate loads. The free edge stresses caused by these loads are next determined. The stress compatibility at the common edges is realized by stress distribution. The resulting stresses shall be referred as case II solution corresponding to an arbitrary rotation of plate 2.

Step 3: Now consider the effect of an arbitrary rotation of plate 3 (Ref. Fig. 2.15). As before,

$$\psi_3 = k_3\ \psi_{30}.$$

The moment introduced at joint 2, is $\dfrac{6EJ_3\Delta_{30}}{b_3^2} = \dfrac{6EJ_3\psi_{30}}{b_3}$

Let the arbitrary rotation ψ_{30} be such that the magnitude of the moment induced is 600 units.

Distribute the moments of 600 units each at joint 2 and 3 by moment distribution procedure. Arrive at the reactions at the joints and apply forces opposite to these at the joints. Resolve these forces into plate loads and compute the free edge stresses. Correct them by stress distribution to secure stress compatibility. The resulting stresses are referred as case III solution.

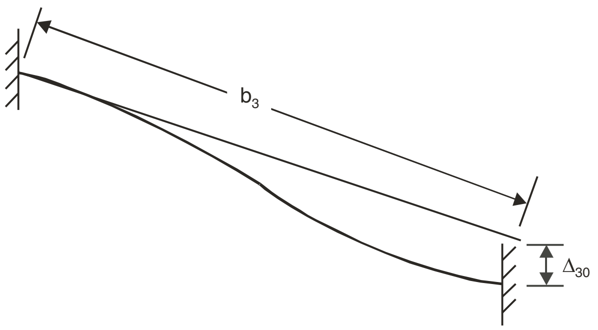


Fig. 12.15

Step 4: The case IV solution correspond to an arbitrary rotation ψ_{40} of plate 4. It is worked out in the same manner as the case III solution. Again, we note that

$$\psi_4 = k_4\ \psi_{40} \qquad \text{...eqn. 12.13}$$

Step 5: The plate deflections Δ_n are next worked out. Δ_n consists of the deflection corresponding to the no rotation solution plus k_2 times the deflection due to case II solution, plus k_3 times the deflection resulting from the case III solution, plus k_4 times the deflection corresponding to the case IV solution.

If edge stresses are known, the corresponding deflections may be calculated as explained below:

Let the stresses at $n-1$ and nth joint be f_{n-1} and f_n respectively as shown in Fig. 12.16. From the figure, it is clear that,

$$\text{bending stress} = f_b = \frac{f_{n-1} - f_n}{2}$$

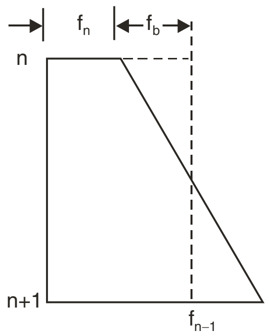


Fig. 12.16

Noting that moment varies in sinusoidal form,

$$EI\frac{\partial^2(\delta)}{\partial y^2} = -M\sin\frac{\pi x}{L}$$

$$\delta = \frac{L^2}{\pi^2}\frac{M}{EI}\sin\frac{\pi x}{L}$$

i.e. mid span deflection, due to moment is

$$\delta = \frac{L^2}{\pi^2}\frac{M}{EI}$$

But $\frac{M}{I} = \frac{f_b}{h/2}$ or $M = \frac{2f_b I}{h_n}$, h is depth of plate.

$$\therefore \quad \delta = \frac{L^2}{\pi^2}\cdot\frac{2f_b I}{EIh_n}$$

$$= \frac{2L^2}{\pi^2}\frac{1}{EIh_n}\left[\frac{f_{n-1}-f_n}{2}\right]$$

$$= \frac{L^2}{\pi^2 Eh_n}\left[f_{n-1}-f_n\right] \qquad \text{...eqn. 12.14}a$$

If closed form solution is used,

$$\delta = \frac{5}{48}\frac{L^2}{Eh_n}\left(f_{n-1}-f_n\right) \qquad \text{...eqn. 12.14}b$$

Total deflection,

$$\Delta_n = \delta_0 + k_2\,\delta_{20} + k_3\,\delta_{30} + k_4\,\delta_{40}$$

Step 6: From the results of step 5, the plate rotations may be calculated using the formula given below:

$$\psi_n = \frac{1}{b_n}\left[\Delta_n\left(\cot\gamma_n + \cot\gamma_{n-1}\right) - \frac{\Delta_{n+1}}{\sin\gamma_n} - \frac{\Delta_{n-1}}{\sin\gamma_{n-1}}\right]$$

and

$$\psi_{n+1} = \frac{1}{b_{n+1}}\left[\Delta_{n+1}\left(\cot\gamma_{n+1} + \cot\gamma_n\right) - \frac{\Delta_{n+1}}{\sin\gamma_{n+1}} - \frac{\Delta_n}{\sin\gamma_n}\right] \qquad \text{...eqn. 12.15}$$

Equating ψ_2, ψ_3 and ψ_4 calculated in Step 6 to $k_2\psi_{20}$, $k_3\psi_{30}$ and $k_4\psi_{40}$ a set of three linear simultaneous equations in unknowns k_2, k_3 and k_4 are obtained. Solve the equations to get k_2, k_3 and k_4.

Step 7: Compute the edge stresses as

$$f_n = f_{n_0} + k_2 f_{n_2} + k_3 f_{n_3} + k_4 f_{n_4} \qquad \text{...eqn. 12.16}$$

Then the shell may be designed.

12.7 STRESS DISTRIBUTION PROCEDURE

To ensure compatibility of stresses at common edges free edge stress distribution is to be carried out. The procedure of stress distribution is similar to moment distribution method with appropriate carry over factor and appropriate distribution factor. In this article, the carry over factor and distribution factors to be used are derived.

The continuity of stresses along edge n common to the plates n and $n + 1$ is ensured by the application of edge shears in longitudinal direction. When plate n and $n + 1$ are regarded as bending independently, free edge stresses f_n and f_{n+1} develop, which are different from each other.

The application of the longitudinal shears at the edges has the effect of correcting the values of f_n and f_{n+1} so that they become equal. To correct those stresses, it is not necessary to solve for edge shears. The correction can be by the stress distribution procedure. Figure 12.17 shows nth and n + 1th plate from one end to mid span. Let T_n be longitudinal edge shear at edge n.

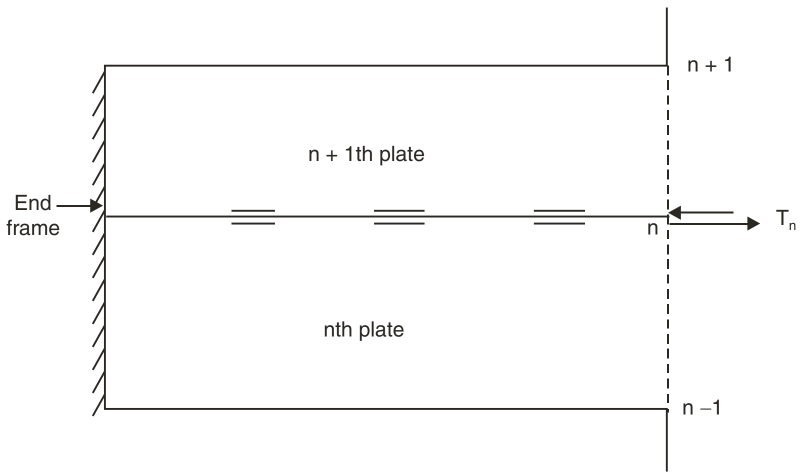


Fig. 12.17

Due to longitudinal shear T_n, stress in plate n at edge n is,

$$= \frac{T_n b_n}{2} \times \frac{1}{I_n} \times \frac{b_n}{2} + \frac{T_n}{h_n b_n}, \text{ where } I_n = \text{moment of inertia of } i\text{th plate}$$

$$= \frac{T_n b_n}{2} \times \frac{12}{h_n b_n^3} \times \frac{b_n}{2} + \frac{T_n}{h_n b_n}$$

$$= \frac{4T_n}{b_n h_n} = \frac{4T_n}{A_n} \quad \text{where } A_n = b_n h_n, \text{ cross-sectional area of } n\text{th plate} \qquad \text{...eqn. 12.17}$$

Similarly, stress at edge $n - 1$ in nth plate

$$= -T_n \cdot \frac{b_n}{2} \times \frac{12}{h_n b_n^3} \times \frac{b_n}{2} + \frac{T_n}{b_n h_n}$$

$$= -\frac{2T_n}{A_n}. \qquad \text{...eqn. 12.18}$$

Stress in n + 1th plate at edge 'n'

$$= -T_n \frac{b_{n+1}}{2} \times \frac{12}{h_{n+1} b_{n+1}^3} - \frac{T_n}{h_{n+1} b_{n+1}}$$

$$= -\frac{4T_n}{h_{n+1} b_{n+1}} = -\frac{4T_1}{A_{n+1}} \qquad \text{...eqn. 12.19}$$

Stress in n + 1th plate at edge n + 1,

$$= \frac{T_n\, b_{n+1}}{2} \times \frac{12}{h_{n+1} b_{n+1}^3} - \frac{T_n}{h_{n+1} b_{n+1}}$$

$$= \frac{2T_n}{h_{n+1} b_{n+1}} = \frac{2T_n}{A_{n+1}} \qquad \text{...eqn. 12.20}$$

If free edge stresses at joint n in plate n is f_n and at joint n in n + 1 plate is f_{n+1} the actual stresses are, in plate n,

$$f_n + \frac{4T_n}{A_n}$$

and in plate n + 1,

$$f_{n+1} - \frac{4T_n}{A_{n+1}}$$

But, the effect of T_n forces are to ensure stress compatibility. Hence,

$$f_n + \frac{4T_n}{A_n} = f_{n+1} - \frac{4T_n}{A_{n+1}}$$

∴ $$4T_n \left(\frac{1}{A_n} + \frac{1}{A_{n+1}} \right) = f_{n+1} - f_n$$

i.e. $$4T_n \frac{A_{n+1} + A_n}{A_n\, A_{n+1}} = f_{n+1} - f_n$$

or $$\frac{4T_n}{A_n} = \frac{A_{n+1}}{A_{n+1} + A_n} \left(f_{n+1} - f_n \right)$$

Thus, the correction to edge stress at n in plate n is $\frac{4T_n}{A_n} = \frac{A_{n+1}}{A_{n+1} + A_n}\left(f_{n+1} - f_n \right)$

Similarly, the correction to edge stress at n in plate n + 1 is

$$-\frac{4T_n}{A+1} = -\frac{A_n}{A_n + A_{n+1}} \left(f_{n+1} - f_n \right) \qquad \text{...eqn. 12.21}$$

Thus, we conclude, if the correction to stress at joint n is $(f_{n+1} - f_n)$, distribute $\frac{A_{n+1}}{A_n + A_{n+1}}$ of it to edge n of plate and $-\frac{A_n}{A_n + A_{n+1}}$ of it to edge n of $(n+1)$th plate. In other words, the distribution factor is $\frac{A_{n+1}}{A_n + A_{n+1}}$ for nth edge of plate n and $\frac{A_n}{A_n + A_{n+1}}$ for nth edge of plate $n + 1$.

From equations 12.17 and 12.18, it is clear that due to T_n stress at nth edge of n plate is $\frac{4T_n}{A_n}$ while in the same plate at other $(n - 1)$th edge is $-\frac{2T_n}{A_n}$. From eqn. 12.19, we find the stress at edge n in $(n + 1)$th plate due to T_n is $\frac{-4T_n}{A_{n+1}}$ while that at other edge $(n + 1)$th is $\frac{2T_n}{A_{n+1}}$. It shows that if corrections are made for stresses at edge n, $-\frac{1}{2}$ of corrected stress should be carried over to other edge. Thus, the carry over factor is $-Y_2$.

Thus, in the stress distribution procedure,

Distribution factors are $\frac{A_{n+1}}{A_n + A_{n+1}}$ and $\frac{A_n}{A_n + A_{n+1}}$ *i.e.* distribution factor is inversely proportional to cross-sectional area and, carry over factor is $-\frac{1}{2}$.

QUESTIONS

1. Discuss the merits and demertis of folded plate roof over shell roofs.
2. Explain briefly structural behaviour of folded plate roofs.
3. List the assumptions made in the analysis of folded plate roofs.
4. Give step by step procedure of analysis of folded plate roof by Whitneys method. Give the recurring equations also.
5. Write short note on Simpson's method of folded plate analysis.
6. Derive the equations of three shears used in folded plate analysis.
7. Derive carry over factor and stress distribution factors to be used in folded plate analysis.

Chapter

13

Introduction to Shells

A shell is a thin curved surface the thickness of which is small compared to the radius and the other two dimensions. Shells are used for roofing large column free areas. Figure 13.1 shows some of the commonly used shell roofs.

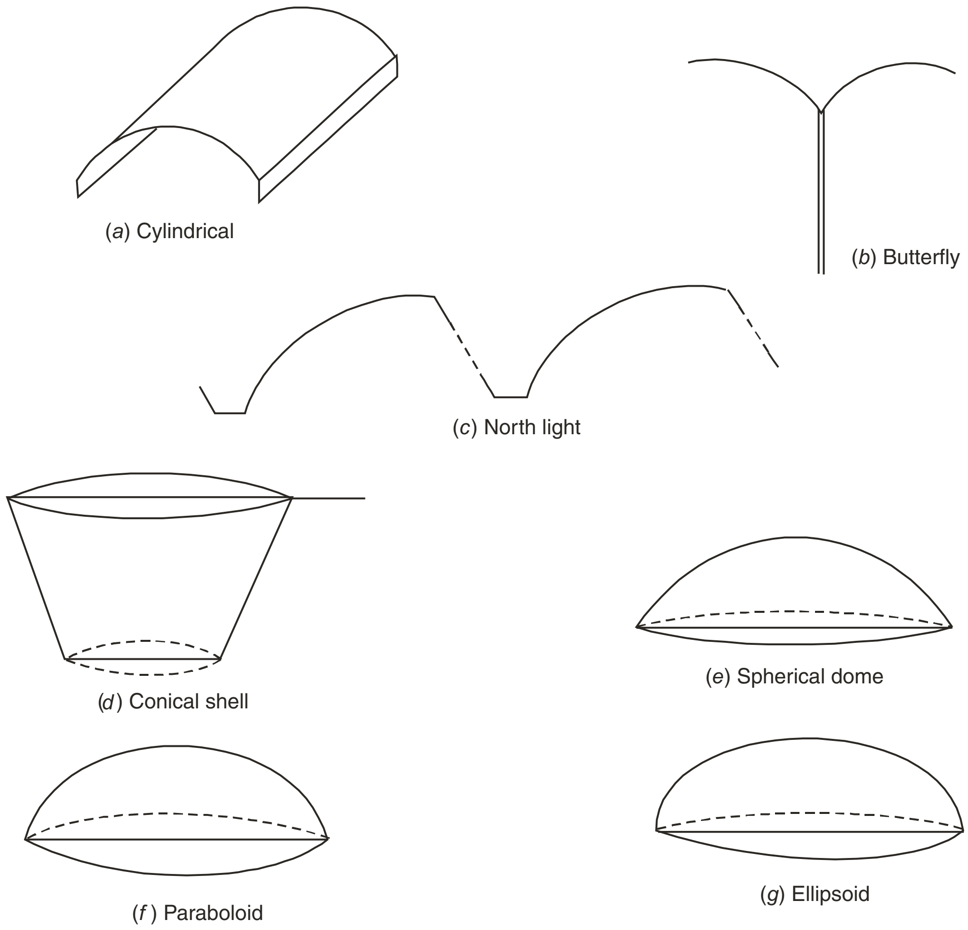
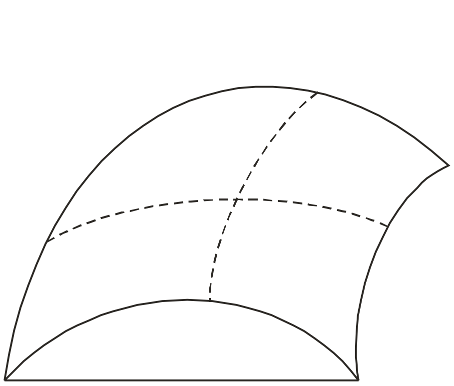
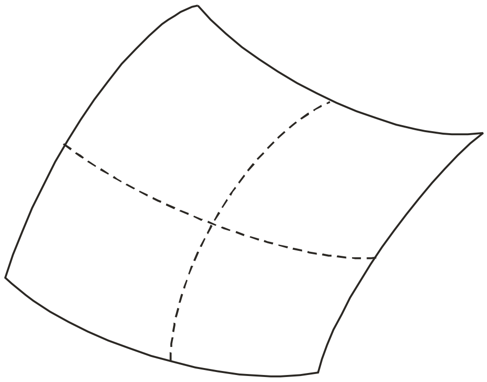


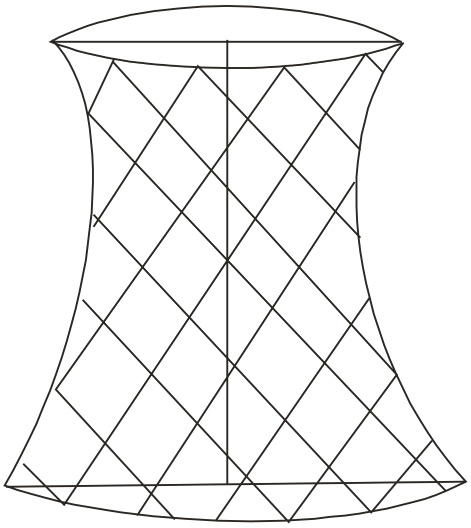
Fig. 13.1 (Contd.)



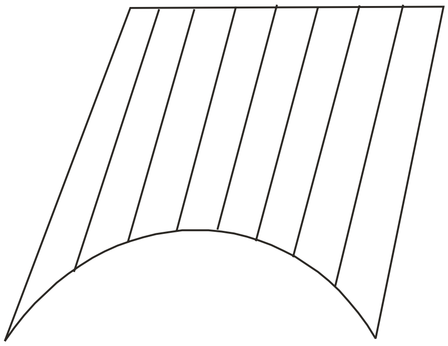
(*h*) Elliptic paraboloid



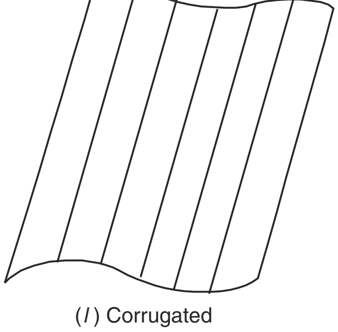
(*i*) Hyperbolic paraboloid



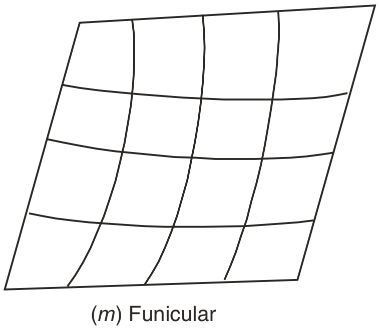
(*j*) Hyperboloid of revolution



(*k*) Conoid



(*l*) Corrugated



(*m*) Funicular

Fig. 13.1

13.1 CLASSIFICATION OF SHELLS

Shells are stressed skin structures *i.e.* the structures in which load transfer is through the skin of structure. In shells, because of their curved shape, bending is reduced to the great extent and load is transferred as compressive force through the membrane of the structure. IS: 2210 gives the classifications of shell on the basis of their shape and the method of generating such shapes. Table 13.1 gives the classification of the shell.

Some of the terms used in classification of shells are defined below:

(*i*) *Shells of Revolution:* The surface generated when a plane curve is rotated about an axis is known as shell of revolution.

(*ii*) *Shells of Translation:* Shells which are obtained by moving one curve parallel to itself along another curve, the planes of the two curves being at right angles to each other.

(*iii*) *Ruled Surfaces:* The surfaces which can be generated entirely by straight lines are known as ruled surfaces. The surface is said to be **singly ruled** if at every point on the shell surface a single straight line entirely lying on the surface can be drawn and it is **doubly ruled** if at every point two straight lines lying entirely on the surface can be drawn.

(*iv*) *Gauss Curvatures:* Gauss curvature is the product of the two principal curvatures at any point on the surface. Thus,

$$\frac{1}{R} = \frac{1}{R_1} \times \frac{1}{R_2}$$

where R is Gauss curvature, R_1 and R_2 are the principal curvatures.

(*v*) *Singly Curved:* A shell is said to be singly curved if its one principal curvature is zero (hence, Gauss curvature is zero). It is also known as **developable** shell, since, such surfaces can be developed from bending plane surface.

(*vi*) *Doubly Curved:* A shell is said to be doubly curved if its Gauss curvature is definite. Such shells are **non-developable.**

(*vii*) *Synclastic shell:* A shell is said to be synclastic if its Gauss curvature is positive.

(*viii*) *Anticlastic shell:* A shell is said to be anticlastic if its Gauss curvature is negative.

13.2 ADVANTAGES AND DISADVANTAGES OF SHELLS

Advantages of shell structures are:

(*i*) Good from the aesthetic point of view

(*ii*) Material consumption is less

(*iii*) Large column free areas can be covered.

(*iv*) Formwork can be removed early.

Disadvantages of shell structures are:

(*i*) Formwork is costly.

(*ii*) Analysis is complicated.

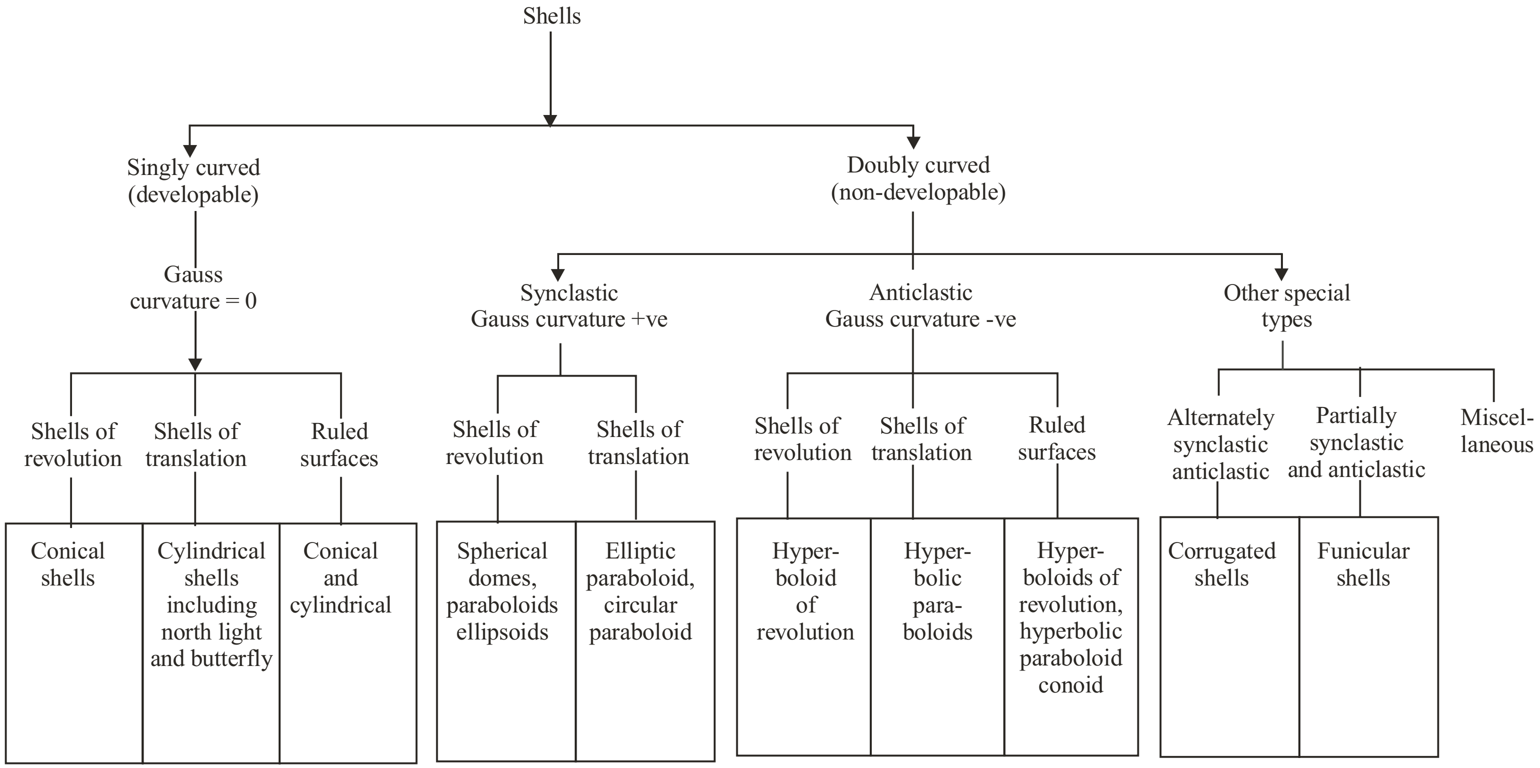
Comments on shell structures: Shell structures become economical if a number of units are to be cast so that repetitive use of formwork is possible. If from aesthetic point of view and from the requirement of large column free area, if an architect prefers shell roof, a structural engineer should design such roofs.

13.3 EFFICIENCY IN THE USE OF MATERIALS

Shells are an example of strength through form as opposed to strength through mass. In a R.C. beam/slab material is subjected to maximum stress only at extreme fibre, that too at the section subject to

(Contd. on p. 162)

Table 13.1 Classification of shells



maximum moment. At all other portion the material is under utilised. In a column most of the material is fully stressed and hence, the material is utilised to maximum extent. That is why we commonly find a column of size 230 × 450 mm support 3 to 4 beams of size 230 × 450 mm. In shells attempt is made to reduce flexure and transfer the load as membrane compression. Hence, in shells material utilisation is improved. It results into use of thinner sections and hence reduction in dead weight. By this means a minimum of material is used to the maximum structural advantage. Shells of double curvatures are among the most efficient of known structural forms. Most shells occurring in nature are doubly curved shells. Shells of eggs, nuts and human skull are common examples.

Thus, in shells strength is due to shape.

13.4 HISTORICAL NOTE

An examination of some of the old places of worship in India and abroad show that shell structures have been in usage even though the theoretical knowledge of their structural behaviour was then meagre. But the modern thin shells are a far cry from the massive masonry domes of the middle ages. A comparison of the relative weights of 17th century Peter Cathedral at Rome and a modern workshop building at Jena, Germany brings home the difference. Both these domes cover an area of 39.6 m diameter.

St. Peter Cathedral at Rome: (Ref. Fig. 13.2)

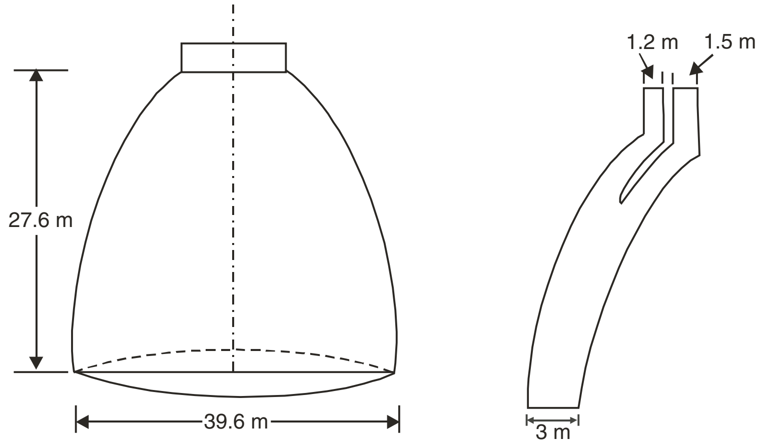


Fig. 13.2 Cathedral at St. Peter, Rome

Total weight = 11000 tonnes

i.e. = 88.42 t/m

It requires heavy foundation.

Now at Schott Workshop, Jena, the dome built is shown in Fig. 13.3.

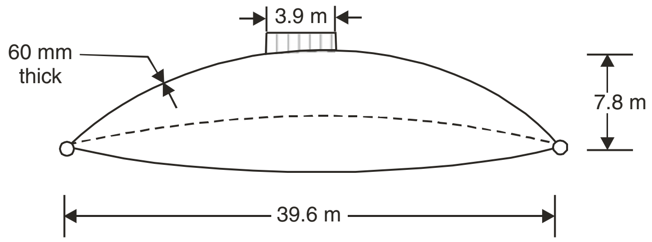


Fig. 13.3 Schott workshop at Jena, Germany

Weight including ring beam = 364 tonnes.

∴ Weight per metre of periphery = 2.93 t/m

One brick wall can take this load.

The above improvement was possible because of the following:

1. R.C.C. is better material compared to masonry.
2. Now there is better understanding of analysis and design.

13.5 A BRIEF LITERATURE REVIEW

The thin reinforced concrete shell had its beginning in Germany in 1920s. Two German engineers—Finsterwalder and Dishinger—were the first two to develop a theoretical analysis applicable to R.C. Cylindrical shells around the year 1930. An early American contribution was a paper by Scharer (1936), which attempted a simplified analysis. Until 1940s, cylindrical shell analysis and construction dominated the research field.

It was only in the mid 1930s and early 1940s that the designers began looking for other forms of shells for roofs. Around 1935, two French engineers, namely, Aimond and Raffaile, published studies on the properties and potentialities of the hyperbolic paraboloid. However, it was the Mexican architect Felix Candola who promoted and popularised the hyperbolic paraboloid for roofing factories, churches, clubs etc.

The conoid had its beginning in France. The one sheet hyperboloid appears to have been first employed for roofing in Germany in the form of Silberkuhl shells. 1950s and 1960s have witnessed an unprecedented spurf at activities in the field of thin shells. Many new forms are continuously experimented upon and added to the growing vocabulary of the shells. With the advent of computers and development of analysis packages, now a days the designers are more confident in going for the shell roofs. They have even tried to optimize shell roof design and have given guidelines for choosing dimensions of various type of shell roof.

13.6 ANALYSIS AND DESIGN

In the analysis of shell roofs, the structure is regarded as homogeneous and isotropic. However, in the design of reinforcement, the concrete is assumed to be cracked and steel is provided to take care of the full direct and diagonal tension.

Shell roofs of complex shapes do not always lend themselves to calculation by analytical means using classical theory. Hence, there is growing trends towards the use of experimental investigations and the use of numerical method like finite element analysis.

To account for the secondary effects like shrinkage and temperature, exact calculations are not available. In tropical countries, the stresses due to temperature changes is sever. The secret of avoiding cracks due to secondary effects lies in the provision of closely spaced small diameter reinforcement.

QUESTIONS

1. Explain the following terms with neat sketches:
   (*i*) Shells of revolution (*ii*) Shells of translation (*iii*) Ruled surfaces.
2. What is Gauss curvature? How do you classify shells based on Gauss curvature values?
3. Give the Indian standard classification of shells.

Chapter

14

Introduction to Cylindrical Shell Roof

Cylindrical shell roof is commonly adopted for covering large column free areas in the factories. As stated in the previous chapter, cylindrical shell is generated when a straight line moves over a curved surface, maintaining its position at right angle to the curve and moving parallel to itself in the projected plane. In this chapter, the various parts and types of cylindrical shell roof are explained and design criteria is presented. A preliminary design procedure, namely, Lundgreens beam theory is presented.

14.1 PARTS OF A SINGLE CYLINDRICAL SHELL ROOF

Figure 14.1 shows a typical single barrel shell and its various parts.

The straight line generating the surface is known as *generator* and the plane curve that guides the generator is known as the *directrix.* The directices that are usually employed are the arc of a circle, the semiellipse, the parabola, the cycloid or the catenary.

A cylindrical shell may or may not be provided with edge beams. The supporting members at the two ends of a shell are known as traverses or end frames. The traverse may be a solid diaphragm, a tied arch, a trussed arch or a rigid frame. It is usually assumed that the shell is simply supported on the traverses. Hence, the traverse should be rigid in its plane but flexible out of plane. This is achieved to a great extent if the frame is deep and thin. The distance between the two adjacent traverses is known as span of the shell and the projection of the directrix is known as chord width.

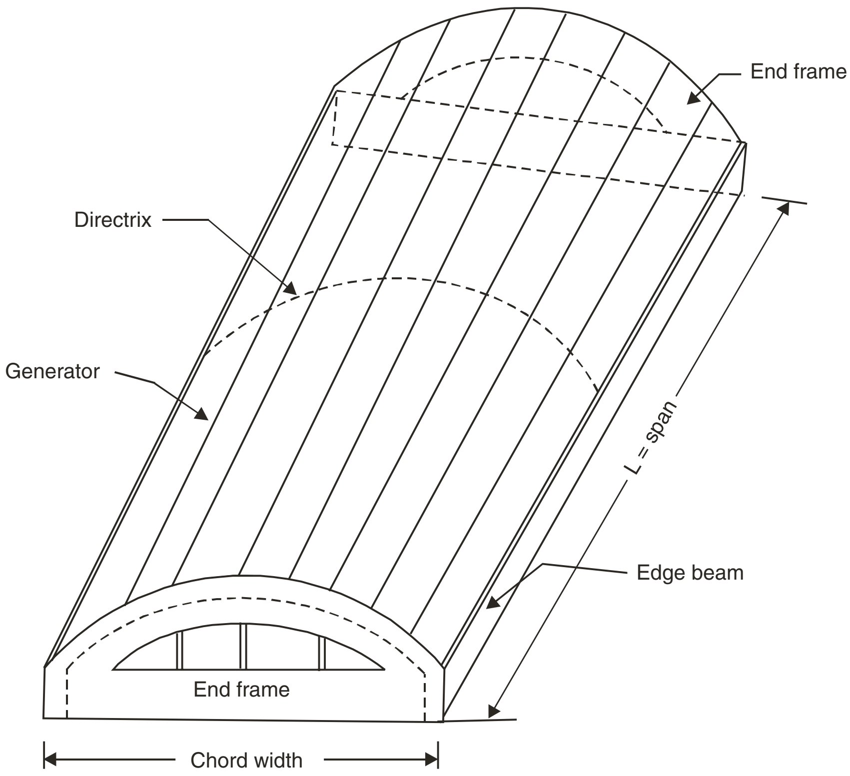
14.2 TYPES OF CYLINDRICAL SHELL ROOFS

The shell roof may be a *single barrel* as shown in Fig. 14.2 (*a*) or it may be a *multiple barrel* as shown in Fig. 14.2(*b*). Usually shells are supported on two end frames. But sometimes they may be built monolithically over more than two end frames as shown in Fig. 14.2(*c*). Such shells are known as *continuous shells.*

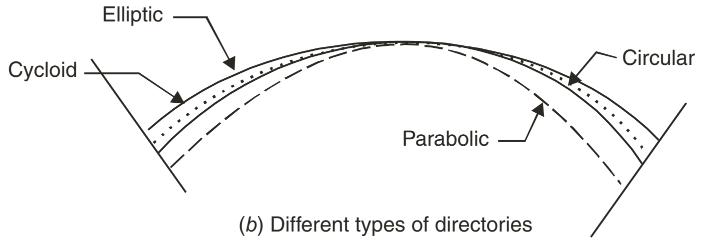
14.3 DESIGN CRITERIA

Indian standard code IS: 2210 recommends the following:

1. **Grade of Concrete:** M:20
2. **Maximum size of the aggregates** is to be restricted to 12 mm to 20 mm depending upon the thickness of the shell.
3. If **chord width** is 8 m to 12 m, maximum span is restricted to 30 m.



(*a*) Parts of cylindrical shell roof

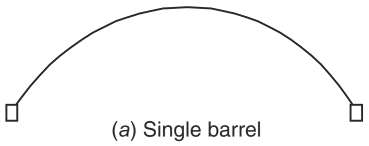


(*b*) Different types of directories

Fig. 14.1

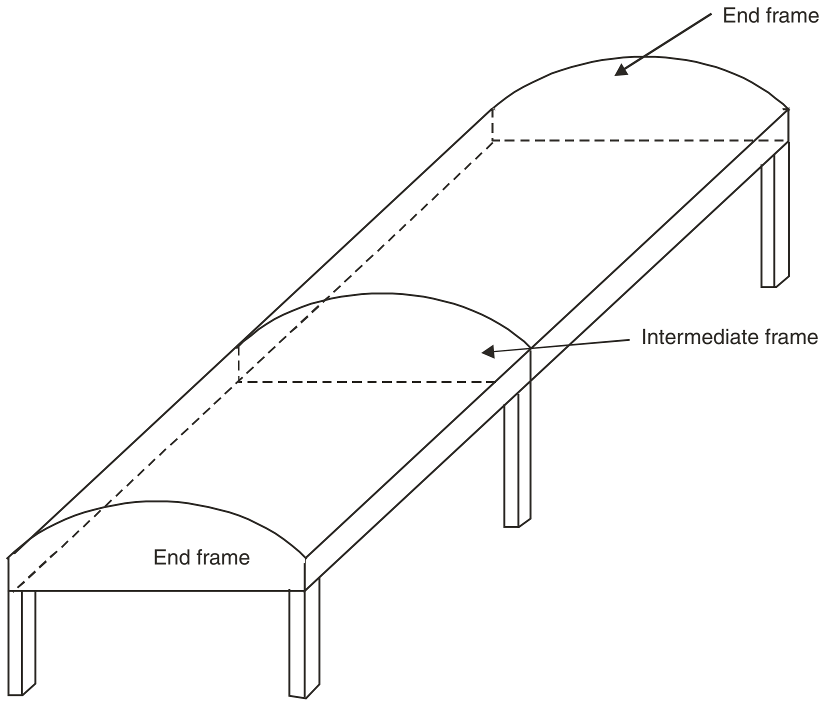
If chord width used is large (say upto 30 m), span should be reduced to 8 to 12 m.
For spans more than 30 m, prestressing of edge beams is necessary.

4. **Height of shell** may be kept 1/12 to 1/6th span—the larger figure applicable to smaller span. However, in case of shells without edge beam, depth shall not be less than 1/10th span. In case of short shells depth should be at least 1/8th of chord width.
5. **Semicentral angle** may be kept between 30° to 40°. Restricting it to 40° has the following advantages:
   (*a*) Wind load effect may be ignored.
   (*b*) During construction back form is not required for concreting.



(*a*) Single barrel

(*b*) Multiple barrel shell



(*c*) Continuous shell

Fig. 14.2

6. **Thickness of shell** should be at least 50 mm to avoid the problems of leakage. Usually thickness selected for long shells is 75 mm while for short shells (more chord width and less span) thickness of 60 mm may be used.

   Near the ends the shell is thickened to 30 percent extra. The distance of thickening from the edge is $0.38\sqrt{rt}$ to $0.78\sqrt{rt}$ where '*r*' is the radius of the shell directrix and '*t*' is the thickness of shell.

   The need for edge thickening are:

   (*a*) Moments in the shells are larger at the ends.

   (*b*) At the edges 3 layers of reinforcement are to be provided namely—longitudinal, transverse and shear. The three layers can be suitably accommodated with required cover only by making edges thicker.

7. **Edge beam** depth is automatically fixed once we select overall depth and semicentral angle. Thickness of the edge beam shall be 2 to 3 times the thickness of the shell.

Example 14.1. Fix up the overall dimensions for a circular cylindrical shell of span 25 m and chord width 10 m.

Solution:

Span L = 25 m

Chord width = 10 m.

Let the semicentral angle be 40°.

$$\text{Then, radius of the shell} = R = \frac{10/2}{\sin 40°} = 7.78 \text{ m}$$

$$\therefore \quad \text{Rise of shell} = R - R\cos 40° = 7.78(1-\cos 40°)$$

$$= 1.82 \text{ m.}$$

Overall depth of shell should be

$$= \frac{1}{12}\text{th to } \frac{1}{6}\text{th span.}$$

$$= \frac{25}{12} \text{ to } \frac{25}{6} \text{ m.}$$

Let us select it as 3.4 m.

∴ Edge Beam Depth = 3.4 − 1.82 = 1.58 m.

Let it be 1.6 m.

Let thickness of shell = 75 mm.

∴ Width of edge beam = 2 to 3 times 75 mm.

Let us select width of edge beam = 200 mm.

Edge thickening

Thickness = 30% extra

= 75 + 0.3 of 75 = 97.6 mm

Let us use 100 mm thickness of edges.

Distance of edge thickening:

$$0.38\sqrt{rt} \text{ to } 0.76\sqrt{rt}$$

i.e. 0.29 m to 0.58 m

Let the edge be thickened to a distance of 600 mm.

Figure 14.3 shows the section selected.

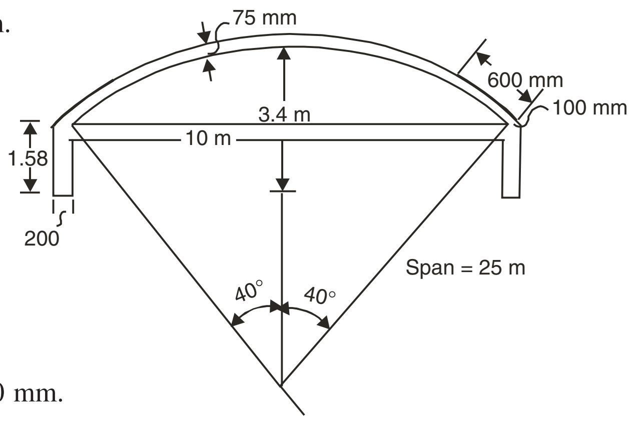


Fig. 14.3

14.4 ANALYSIS

Various methods of analysis found in literature may be listed as:

(*i*) Beam theory
(*ii*) Membrane theory
(*iii*) Bending theory
(*iv*) Finite element method

Beam theory is explained in this chapter. Membrane theory and Bending theory are taken in the next two chapters. Finite element method is treated as out of the scope of this book. For this method, readers may refer the books on the finite element analysis.

14.5 BEAM THEORY

This theory can be used for the analysis of cylindrical shells with span more than three times the chord width. This theory is based on the assumption that plane cross-section remains plane even after deformation. In other words, it assumes longitudinal stress varies linearly across the depth of section. This assumption holds good fairly well for long shells. Hence, one can use it for the analysis of such shells.

This theory is relevant even today to study because it gives feel of the structural behaviour and the designer can avoid mixing up of the sign conventions in the sophisticated analysis and committing the blunder. This method has the *following advantages:*

1. It brings shell analysis within the reach of those who are not familiar with the technique of advanced mathematics.
2. Shells with non-circular directices can be dealt with easily.
3. Shells with non-uniform thickness also can be handled without much difficulty.
4. Structural action of the shell is easily visualised.

This theory divides the shell action into

(*i*) beam action

(*ii*) arch action

14.5.1 Beam Action

A long cylindrical shell is analysed as a beam with curved cross-section, supported on end frames. To start with entire concrete section is assumed effective to determine the longitudinal stresses and shear stresses. In the design wherever tension is found, concrete is treated as ineffective and steel is provided to take the tension. Consider the typical cross section of the shell shown in Fig. 14.4.

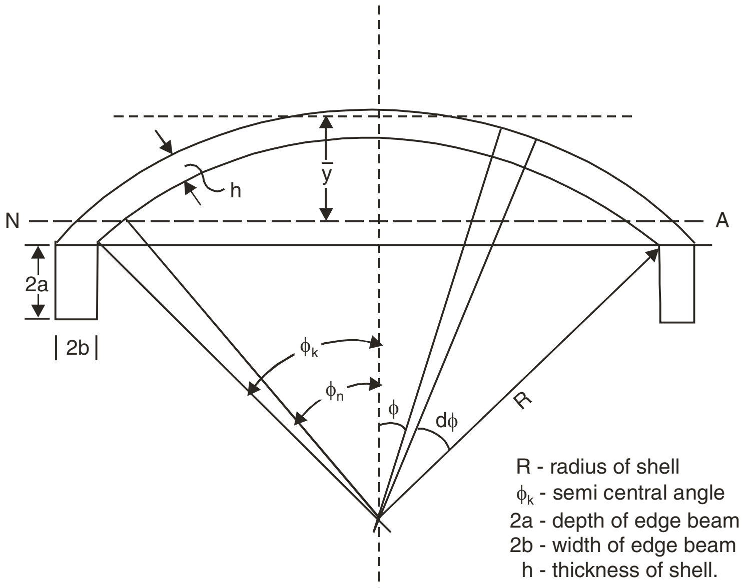


Fig. 14.4

Let the neutral axis be at a distance $\bar{y}$ from the horizontal axis through crown. Then,

$$\bar{y} = \frac{\text{Moment of area about the horizontal axis at crown}}{\text{Total area of shell}}$$

$$= \frac{\left[2a \times 2b\left\{R\left(1-\cos\phi_k\right)+a\right\} + \int_0^{\phi_k} Rd\phi h\left(R - R\cos\phi\right)\right] \times 2}{\left[2a \times 2b + hR\phi_k\right] \times 2}$$

$$= \frac{4ab\left\{R\left(1-\cos\phi_k\right)+a\right\} + R^2 h\left\{\phi_k - \sin\phi_k\right\}}{4ab + hR\phi_k} \quad \text{...eqn. 14.1}$$

If ϕ_n is the semicentral angle made by the neutral point on shell as shown in the figure, then it can be determined from the relation:

$$R\left(1-\cos\phi_n\right) = \bar{y}\,. \quad \text{...eqn. 14.2}$$

Then the moment of inertia of the section about neutral axis '*I*' is given by

I = Moment of inertia of the edge beams + Moment of inertia of the shell.

$$= \left[\frac{1}{12} \times 2b(2a)^3 + 2a \times 2b\left\{R\left(1-\cos\phi_k\right) - \bar{y} + a\right\}^2 + \int_0^{\phi_k} R\,d\phi h\,R^2\left(\cos\phi - \cos\phi_n\right)^2\right] \times 2$$

Using the expression, $\sigma = \frac{M}{I}y,$ stresses at any required depth can be found, in which *M* is the moment. It is maximum at centre of span as it is a simply supported beam of span *L* subjected to uniformly distributed load *w i.e.* maximum moment is $= \frac{wL^2}{8}.$

Obviously bending stress is zero at neutral axis and is having maximum compressive at crown point in midspan. This stress should be checked for bending compression. To take care of tensile stresses provide longitudinal reinforcement to take complete tension. It may be observed that only a small portion of shell (that is below *N-A*) and edge beams are in tension. Tensile force in the edge beam may be considered equal to the tensile stress at mid depth of the edge beam multiplied by the area of the edge beam. Reinforcement found is provided at closer interval in the lower portion and spacing is uniformly increased towards the top.

The beam with this curved shape is to be designed to withstand shear stresses also. Maximum shear force is equal to 1/2 the total load and occurs near end frames.

$$V = \frac{wL}{2}.$$

Maximum shear stress $= \frac{V}{bI}\left(a\bar{y}\right),$ where

b = width of shell at neutral axis = $2h$

$a\bar{y}$ = moment of area above neutral axis about the neutral axis

$$= 2\int_0^{\phi_n} R d\phi h \left(R\cos\phi - R\cos\phi_n\right)$$

$$= 2R^2 h\left[\sin\phi - \phi\cos\phi_n\right]_0^{\phi_n}$$

$$= 2R^2 h\left[\sin\phi_n - \phi_n\cos\phi_n\right]$$

Shear reinforcement is provided diagonally at corners to a maximum length of $L/4$.

14.5.2 Arch Action

Consider the freebody diagram of a shell of unit length as shown in Fig. 14.5.

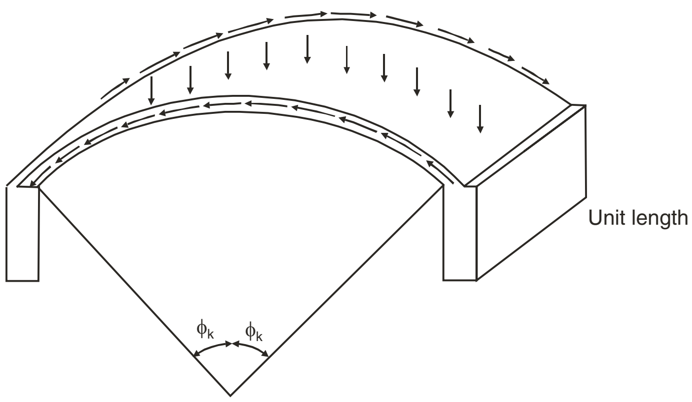


Fig. 14.5 Arch action

The equilibrium of the unit length of shell is maintained by two sets of forces, namely the load w per unit length of shell and the specific shear $\delta = \frac{w}{bI}\left(a\bar{y}\right)$ where the specific shear is defined as difference in shear forces between the two edges of the unit length of shell.

The specific shear acting at any point in the direction of the tangent to the shell arch may be resolved into its horizontal and vertical components. It is clear that the sum of the vertical components of specific shear balances the vertical load on the arch and the sum of the horizontal components will be zero.

If it is a single barrel shell, it is obvious that the elemental shell arch do not develop any restraining forces or moments at its ends. Hence, we have a statically determinate free arch for the analysis.

If it is an intermediate shell of a multiple shell, the ends will behave like fixed ends. Hence, fixed arch analysis is necessary. The column analogy or the elastic centre method is suitable for the analysis of fixed archies.

Example 14.2. Design a reinforced concrete circular shell with the following particulars.

Radius	$R = 3$ m
Span	$L = 15$ m
Semi central angle	$\phi_k = 60°$
Thickness of shell	$h = 75$ mm

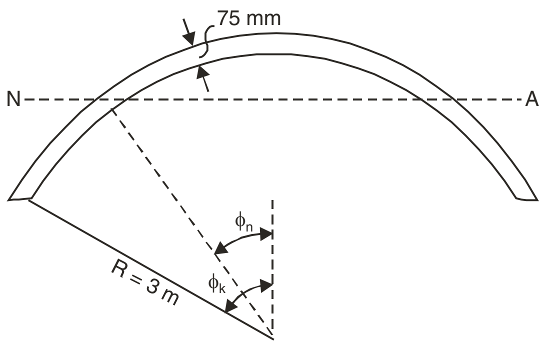


Fig. 14.6

Loads

Self weight = 0.075 × 1 × 1 × 25 = 1.875 kN/m^2

Water proofing cover and occasional live load = 1 kN/m^2

∴ Total load = 2.875 kN/m^2

Weight per metre run of shell

$$w = 2.875 \times R \times 2\phi_k = 2.875 \times 3 \times 2 \times \frac{\pi}{3} = 18.06 \,\text{kN}/\text{m}^2$$

Maximum bending moment

$$\frac{wL^2}{8} = 18.06 \times \frac{15^2}{8} = 508 \text{ kN-m}$$

Since, edge beams are not provided, $a = b = 0$

Referring to Fig. 14.6,

$$\therefore \quad \bar{y} = \frac{R^2 h(\phi_k - \sin\phi_k)}{hR\phi_k} = \frac{R(\phi_k - \sin\phi_k)}{\phi_k}$$

$$= \frac{3\left[\frac{\pi}{3} - \sin 60\right]}{\frac{\pi}{3}} = 0.519 \text{ m}$$

∴ ϕ_n is obtained by the relations

$$\bar{y} = R - R\cos\phi_n$$

$$0.519 = 3(1 - \cos\phi_n)$$

$$\therefore \quad \phi_n = 34.2°$$

$$I = 2\int_0^{\phi_k} Rd\phi\, h\, R^2 (\cos\phi - \cos\phi_n)^2\, d\phi$$

$$= 2R^3 h \int_0^{\phi_k} \left[\frac{1 + \cos 2\phi}{2} - 2\cos\phi\cos\phi_n + \cos^2\phi_n\right] d\phi$$

$$= 2R^3h\left[\frac{\phi_k + \frac{1}{2}\sin 2\phi_k}{2} - 2\cos\phi_n \sin\phi_k + \phi_k \cos^2\phi_n\right]$$

$$= 2\times 3^2 \times 0.075\left[\frac{\frac{\pi}{3} + \frac{1}{2}\sin(2\times 34.2)}{2} - 2\cos 34.2\times\sin 60 + \frac{\pi}{3}\cos^2 34.2\right]$$

$$= 0.0968\,\text{m}^4 = 0.0968\times 10^{12}\ \text{mm}^4$$

∴ Compressive stress at crown

$$= \frac{M}{I}\bar{y} = \frac{508\times 10^6}{0.0968\times 10^{12}}\times 519 = 2.723\ \text{N/mm}^2 < 8.5\,\text{N/mm}^2$$

Rise of shell $= R - R\cos 60 = 3(1-\cos 60) = 1.5$ m

∴ The distance of lowest point from *NA* = 1.5 – 0.519 = 0.981 m = 981 mm

∴ Maximum Tensile Stress in shell

$$= \frac{508\times 10^6}{0.0968\times 10^{12}}\times 981 = 5.148\,\text{N/mm}^2$$

∴ Tensile force per metre run of shell = 5.148 × 75 × 1000 = 386116 N

∴ $$A_{st} = \frac{386116}{150} = 2574\ \text{mm}^2$$

Provide 16 mm bars at 75 mm c/c. This may be changed to 12 mm bars at 200 mm c/c near neutral axis and is maintained in the compression zone.

Design for Shear

Maximum shear force $= 18.064\times\frac{15}{2} = 135.48$ kN

At neutral axis

$$a\bar{y} = 2R^2h[\sin\phi_n - \phi_n\cos\phi_n]$$

$$= 2\times 3^2\times 0.075\left[\sin 34.2 - \frac{34.2\times\pi}{180}\cos^2 34.2\right]$$

$= 0.09234\ \text{m}^3 = 0.09234\times 10^9\ \text{mm}^3$

∴ Shear strees at neutral axis

$$q = \frac{V}{bI}(\bar{a}y)$$

$$= \frac{135.48\times 1000}{2\times 75\times 0.0968\times 10^{12}}\times 0.09234\times 10^9$$

$= 0.86\ \text{N/mm}^2$

$$\tau_v = 1.5 \times 0.86 = 1.29 \text{ N/mm}^2 < \tau_c \text{ max}$$

Hence, shear reinforcement can be provided to take care of shear.

$$A_{st} \text{ provided at } N\text{-}A = \frac{\frac{\pi}{4} \times 12^2}{200} \times 1000 = 565.48 \text{ mm}^2 \text{ per meter width}$$

$$\text{Percentage of reinforcement } = \frac{A_{st}}{h \times 1000} \times 100 = \frac{565.48}{75 \times 1000} \times 100 = 0.754$$

$$\tau_c = 0.57 \text{ N/mm}^2$$

Thus $\tau_c < \tau_v < \tau_{c \text{ max.}}$

Hence, shear reinforcement is to be provided.

Shear force per metre width of shell

$$V_u = \tau_v \times h \times 1000 = 1.29 \times 75 \times 1000 = 96750 \text{ N.}$$

$$\therefore \quad V_{us} = 96750 - \tau_c\, h \times 1000$$

$$= 96750 - 0.57 \times 75 \times 1000 = 54000 \text{ N}$$

using 8 mm bars [note it is single legged is shells] at 45°

$$s = \frac{\frac{\pi}{4} \times 8^2 \times 0.87 \times 415 \times 1000 [\sin 45° + \cos 45°]}{54000} = 475 \text{ mm}$$

Hence, provide nominal shear reinforcement of 8 mm at 200 mm *c/c*.

Arch Action

Load per unit length of shell, w = 18.06 kN

$$\therefore \text{ Specific shear} \qquad q_s = \frac{18.06 \times 1000}{(2h) I} (\bar{a}y)$$

$$= \frac{18060}{2 \times 75 \times 0.1613 \times 10^{12}} 2R^2 h [\sin\phi - \phi \cos 60]$$

$$= \frac{18060 \times 2 \times (3000)^2 \times 72}{2 \times 75 \times 0.1613 \times 10^{12}} [\sin\phi - 0.5\phi]$$

$$= 1007.68 \, [\sin \phi - 0.5 \, \phi]$$

The arch is divided into a suitable number of parts for further calculation. In this case, let us divide it into 12 parts at 10° intervals as shown in Fig. 14.7. The specific shear at centre of each part is calculated using the above expression and is shown in Table 14.2. Specific shear force

$$T_s = q_s A = q_s \times \frac{10}{180} \pi \times 75$$

$\therefore$ Net vertical force, $V = w_s - T_s \sin \phi$, where w_s is the vertical load on each part.

$$w_s = \frac{18.06}{12} = 1.505 \text{ kN} = 1505 \text{ N}$$

and net horizontal force $\qquad H = T_s \cos \phi$

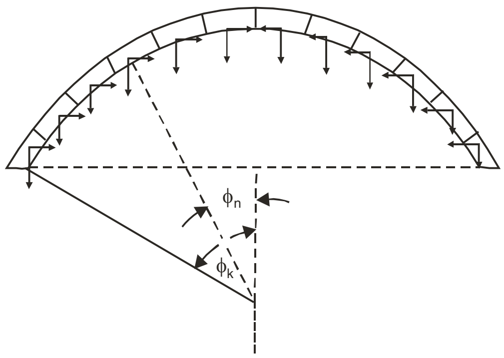


Fig. 14.7

Referring to Fig. 14.7,

Moment at crown = $\Sigma(V R \sin \phi + H R (1 - \cos \phi))$

Moment at ϕ = Moment at crown $- \Sigma(V R \sin \phi + H R (1 - \cos \phi))$

In the expression for moment at ϕ summation is to be considered for the points above the one under consideration. Table 14.1 shows these calculations.

Table 14.1 Calculation Table

Section	*φ*		*q_s in N/mm²*	*T_s in N*	*V in N*	*H in N*	*VR sin φ + HR(1 − cos φ) in N-m*	*M in N-m*
	Deg.	*Radians*						
1	5	0.0873	43.8	574.1	1455	572.0	386.96	340.5
2	15	0.2618	128.9	1687.3	1068.3	1630.0	996.1	–655.5
3	25	0.4363	206.0	2696.8	365.3	2444.0	1150.1	–1805.6
4	35	0.6109	270.2	3537.0	–523.7	2897.0	670.6	–2476.3
5	45	0.7854	316.8	4147.2	–1427.5	2933.0	–451.0	–2025.3
6	55	0.9599	341.8	4474.0	–2159.9	2566.0	–2025.5	0
							Σ 727.5	

∴ Moment at crown 727.5

Moment for design = 2476.3 N-m

$$M_u = 1.5 \times 2476.3 = 3714.45 \text{ N-m}$$

Taking effective depth = 75 − 25 = 50 mm

$$3714.45 \times 10^3 = 0.87 \times 415 \times A_{st} \times 50\left(1 - \frac{A_{st}}{1000 \times 50} \times \frac{415}{25}\right)$$

$$205.8 = A_{st}\left(1 - \frac{A_{st}}{3012}\right)$$

or

$$A_{st}^2 - 3012 A_{st} + 205.8 \times 3012 = 0$$

$$A_{st} = 222 \text{ mm}^2$$

Provide 8 mm bars at $\dfrac{\dfrac{\pi}{4} \times 8^2}{222} \times 1000 \approx 220 \text{ mm c/c}$

The bars are to be provided on appropriate side as shown in Fig. 14.8.

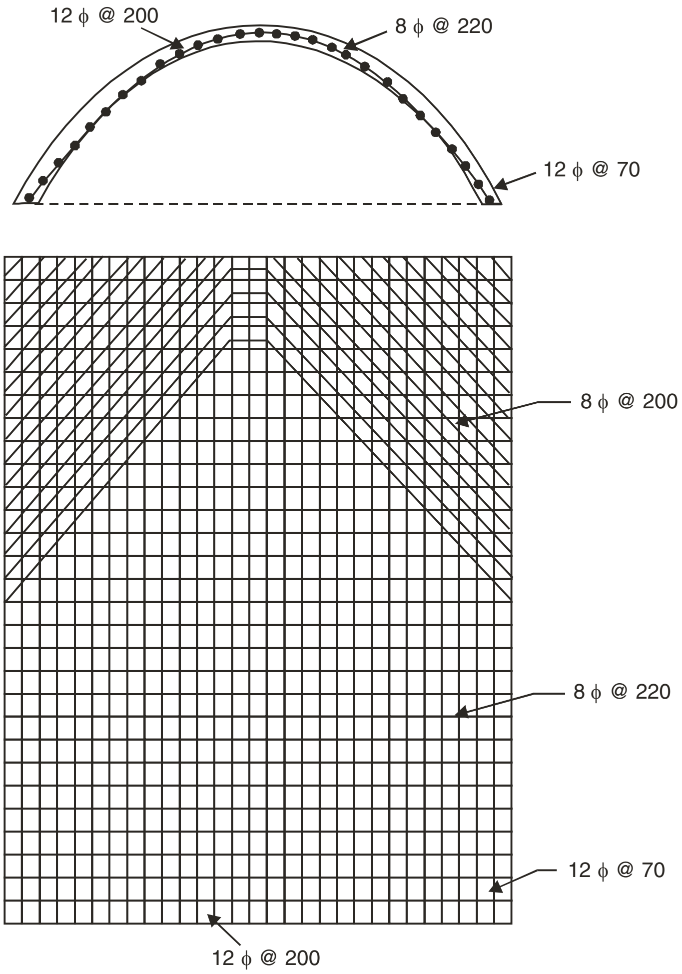


Fig. 14.8

Example 14.3. An intermediate shell of a multiple cylindrical shell roof of span 25 m is to be built. Each shell unit has a chord width of 8 m. Fix up overall size of the shell and design longitudinal and shear reinforcement. Explain how analysis for arch action will be carried out and then give typical reinforcement details.

Solution.

$L = 25$ m Chord width = 8 m

Let $\theta_k = 40°$.

Then
$$R = \frac{4}{\sin 40°} = 6.22 \text{ m}$$

Rise = $R - R \cos 40° = 6.22(1 - \cos 40°)$
= 1.46 m.

The overall depth of shell should be between $\frac{1}{12}$ th to $\frac{1}{6}$ th span *i.e.* in this case it is to be between $\frac{25}{12}$ to $\frac{25}{6}$ m. Let us select overall depth = 3.0 m.

∴ Edge beam depth = 3.0 − 1.46 = 1.54 m. Let the thickness of shell be = 75 mm.

∴ Width of edge beam = 2 to 3 times 75 mm. Let us select width of edge beam = 200 mm.

Figure 14.9 shows the typical section selected.

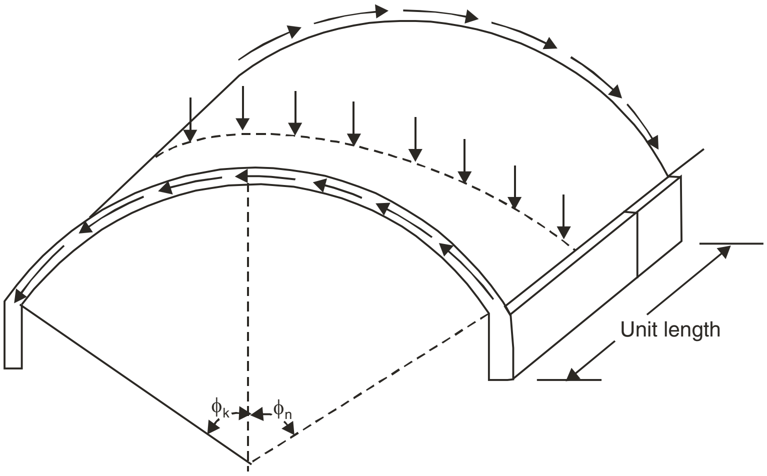


Fig. 14.9

Half of the edge beam is assumed to take care of shell on either side

∴ $2b = 0.1$ or $b = 0.05$ m

$2a = 1.54$ m or $a = 0.77$ m

∴ Distance of *N-A* from the crown

$$\bar{y} = \frac{4ab\left[R\left(1-\cos\phi_k\right)+a\right]+R^2h\left(\phi_k-\sin\phi_k\right)}{4ab+Rh\phi_k}$$

$$= \frac{4\times 0.77\times 0.05\left[6.22\left(1-\cos 40°\right)+0.77\right]+6.22^2\times 0.075\left[40\times\frac{\pi}{180}-\sin 40°\right]}{4\times 0.77\times 0.05+6.22\times 0.075\times 40\times\frac{\pi}{180}}$$

= 1.05 m

Let the intersection of neutral axis with shell make angle ϕ_n with vertical through crown [Ref. Fig. 14.10]. Then

$$R(1 - \cos\phi_n) = 1.05$$

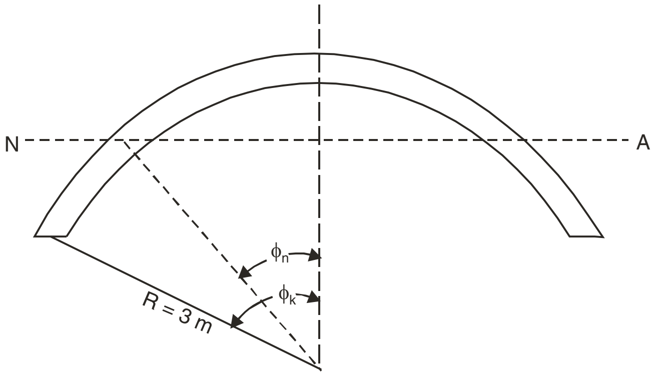


Fig. 14.10

$$6.22(1 - \cos\phi_n) = 1.05$$

$$\therefore \quad \phi_n = 33.76°$$

$$I = \frac{1}{12} \times 0.1 \times 1.54^3 + 1.6 \times 0.1\{6.22(1 - \cos 40°) - 1.05 + 0.77\}^2 \times 2$$

$$+2 \times 6.22^3 \times 0.075\left[\frac{\frac{40 \times \pi}{180} + \frac{1}{2}\sin 80°}{2} - 2\cos 33.765° \times \sin 40° + \frac{40 \times \pi}{180}\cos^2 33.765°\right]$$

$$= 0.828276 \text{ m}^4 = 0.828276 \times 10^{12} \text{ mm}^4$$

Loads

Self weight of shell = 0.075 × 25 = 1.875 kN/m^2
Occasional live load plus finishing load = 1 kN/m^2
Total load on shell = 2.875 kN/m^2

$$\therefore \text{ Load per metre run of shell} = 2.875 \times 80 \times \frac{\pi}{180} \times 6.22$$

$$= 24.97 \text{ kN/m}$$

Weight of edge beams = 0.1 × 1.54 × 25 × 2 = 7.7 kN/m
Valley finishing and weight of edge finishing = 1.5 kN/m
∴ Total load W = 34.17 kN/m

Design for Beam Action

$$\text{Maximum moment } M = \frac{wL^2}{8} = 34.17 \times \frac{25^2}{8} = 2669.531 \text{ kN-m}$$

$$\text{Maximum shear } V = \frac{wL}{2} = 34.17 \times \frac{25}{2} = 427.125 \text{ kN}$$

∴ Compressive stress at crown

$$\sigma_c = \frac{M}{I} y = \frac{2669.531 \times 10^6}{0.828276 \times 10^{12}} \times 1.05 \times 10^3$$

$$= 3.384 \text{ N/mm}^2 < 7.0 \text{ N/mm}^2$$

where 7.0 N/mm^2 is permissible stress in bending compression in M:20 concrete.

Hence, the section selected is safe in bending compression.

Tensile Stress at Centre of Edge Beam:

Distance of centre of edge beam from neutral axis:

$$y = R\,(1 - \cos \phi_k) + a - R\,(1 - \cos \phi_n)$$
$$= 6.22\,(1 - \cos 40°) + 0.77 - 6.22\,(1 - \cos 33.765°)$$
$$= 1.176 \text{ m}$$

Tensile stress in edge beam at its centre

$$\sigma_{av} = \frac{M}{I} y = \frac{2669.531 \times 10^6}{0.828276 \times 10^{12}} \times 1.176 \times 10^3 = 3.79 \text{ N/mm}^2$$

Total tensile force in edge beam

$$= 3.79 \times 1.54 \times 1000 \times 0.1 \times 1000$$
$$= 583698 \text{ N}$$

$$\therefore \quad A_{st} \text{ reqd} = \frac{583698}{150} = 3891 \text{ mm}^2$$

Using 28 mm bars, number of bars required

$$= \frac{3891}{\frac{\pi}{4} \times 28^2} = 7$$

Provide 7 bars of 28 mm diameter Fe-415 steel on each face of edge beam, since, another half of the beam has to take care of loads from the adjoining shell.

Lower portion of shell is also in tension. Height of this portion

$$= R\left(1 - \cos \phi_k\right) - \bar{y}$$
$$= 6.22\,(1 - \cos 40°) - 1.05$$
$$= 0.405 \text{ m}$$

Maximum tension in this tensile portion of shell is at the junction with edge beam.

This stress $= \frac{M}{I} \times y$

$$= \frac{2669.531 \times 10^6}{0.828276 \times 10^{12}} \times 0.405 \times 1000$$
$$= 1.305 \text{ N/mm}^2$$

Thickness of shell at junction is 30% more than shell thickness.

i.e. Thickness = 1.3 × 75 = 100 mm (say)

$$\therefore \text{ Tensile steel required } = \frac{1.305 \times 100 \times 1000}{150} = 870 \text{ mm}^2 \text{ per metre length.}$$

∴ Provide 10 mm bars at 90 mm centre to centre in the lower portion of shell. After providing 4 such bars, in the remaining portion nominal reinforcement of 10 mm @ 200 mm c/c may be provided [Ref. Fig. 14.11].

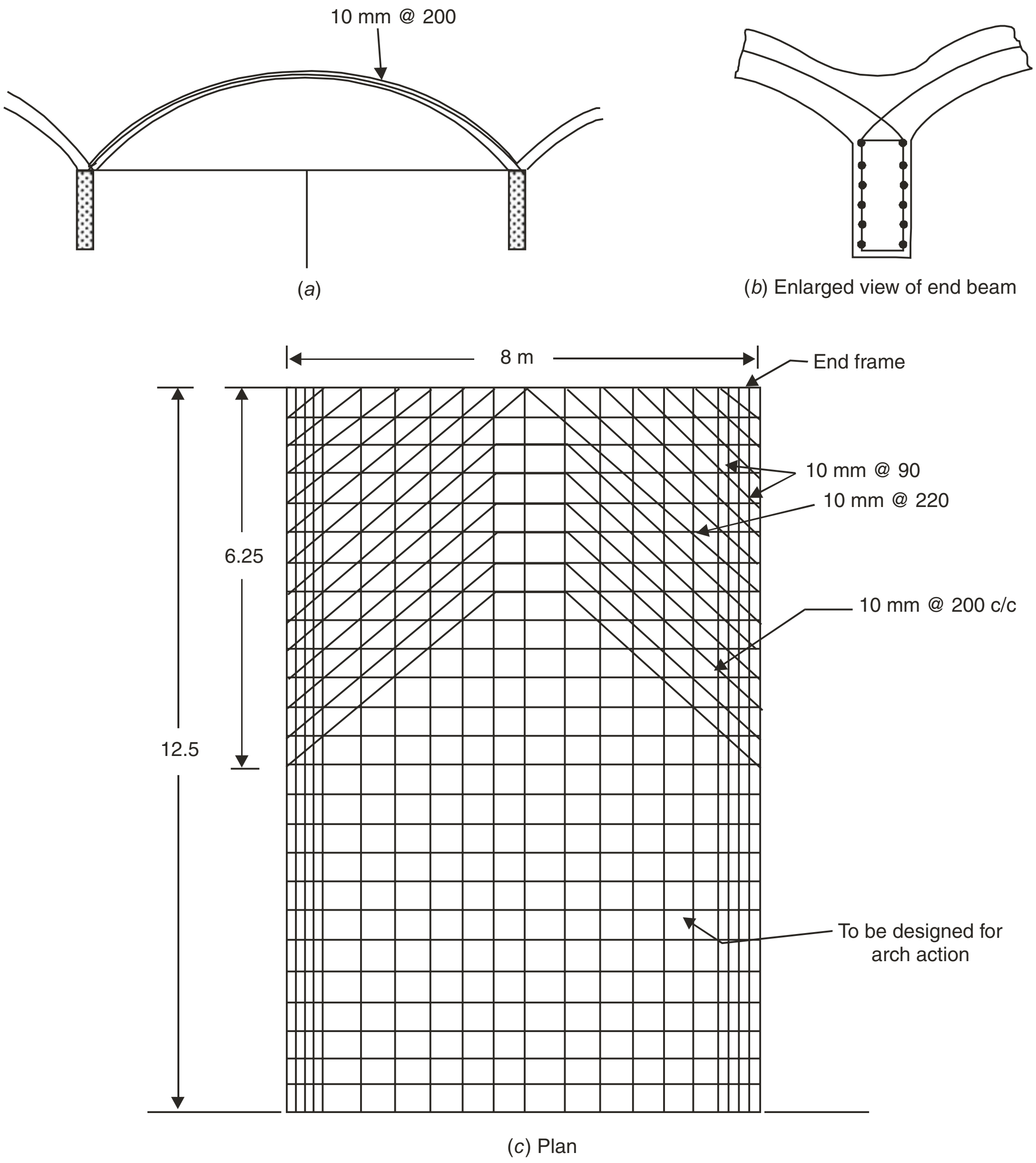


Fig. 14.11 Typical Reinforcement in cylindrical shell

Design for Shear

Shear stress is maximum at the neutral axis in the end section. Here

$$a\overline{y} = 2R^2h\left[\sin\phi_n - \phi_n \cos\phi_n\right]$$

$$= 2 \times 6.22^2 \times 0.075\left[\sin 33.765° - \frac{33.765 \times \pi}{180}\cos 33.765\right]$$

$= 0.382318 \text{ m}^3$

$= 0.382318 \times 10^9 \text{ mm}^3$

∴ Shear stress at N-A

$$q = \frac{V}{bI}(a\bar{y})$$

$$= \frac{427.125 \times 1000}{(2 \times 75) \times 0.828276 \times 10^{12}} \times 0.382318 \times 10^9$$

$= 1.314 \text{ N/mm}^2$

$$A_{st} \text{ provided} = \frac{\pi/4 \times 10^2}{200} \times 1000 = 392.7 \text{ mm}^2$$

$$\therefore \quad \% \text{ steel} = \frac{392.7}{75 \times 1000} \times 100 = 0.523$$

∴ $\tau_c = 0.48 \text{ N/mm}^2$. [for M: 200 mm].

Hence, shear reinforcement is to be designed.

Factored shear force per metre length of shell

$V_u = 1.5 \times 1.314 \times 1000 \times 75 = 147825 \text{ N}$

∴ $V_{us} = V_u - \tau_c bd$

$V_{us} = 147825 - 0.48 \times 75 \times 1000 = 111825 \text{ N}$

Using 8 mm stirrups at 45°,

$$S_v = \frac{0.87 f_y A_{sv} d}{V_{us}}(\sin\alpha + \cos\alpha)$$

$$= \frac{0.87 \times 415 \times \frac{\pi}{4} \times 8^2 \times 1000(\sin 45° + \cos 45°)}{111825}$$

$= 229 \text{ mm}.$

Provide 8 mm bars at 220 mm c/c.

Design for Arch Action

The intermediate shell acts as a fixed arch. The specific shear may be found at centres of several equal parts. Horizontal and vertical loads due to specific shear may be determined for each part. The vertical downward load due to self weight of each segment may be calculated. Then the arch may be analysed by elastic centre method to get transverse moments at the centre of each part. Then the reinforcement required to resist the transverse moment may be determined. The transverse moment is –ve near edges and positive at crown. Hence, the reinforcement is on upper side at edges while it is on lower side at crown. However, since the thickness is small sometimes one layer on upper side and another layer on lower side are also provided.

Figure 14.11 shows the details of reinforcement.

QUESTIONS

1. Draw a neat sketch of a single barrel shell and indicate its various parts.
2. List the Indian standard recommendations for the design of cylindrical shell and fix up the overall dimensions for a shell to cover an area 24 m × 8 m.
3. What are the advantages of beamy theory for the analysis of cylindrical shells? What are its limitations?
4. Briefly explain Leudgreen's beam theory for the analysis of a shell with edge beam giving necessary equations for beam action. Arch action may be explained qualitatively.
5. Design longitudinal and shear reinforcement in a circular cylindrical shell of span 24 m and chord width 8 m. Use beam theory.
6. Draw neatly typical reinforcements in a cylindrical shell roof with edge beam.

Chapter

15

Membrane Analysis of Cylindrical Shell Roofs

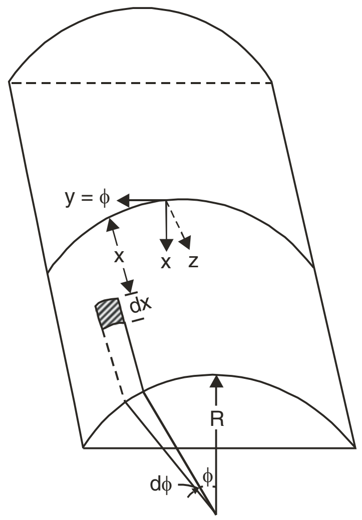
In the membrane theory, the shell is idealised as a membrane incapable of resisting bending stresses. In other word, in this theory, bending of element of a shell is ignored and it is treated as though it is under the forces through the skin of the shell structure.

In case of cylindrical shells bending of the element is not negligible. To get reasonably good behaviour of shell, the bending should be considered. Hence, one should design cylindrical shell roofs only after analyzing by bending theory. However, we study the membrane theory first because of the following reasons.

1. It is useful in many practical cases in gaining some insight into the structural behaviour of a shell.
2. We see later that the membrane theory can be used as a particular integral in the bending theory.

15.1 EQUATIONS OF EQUILIBRIUM

Figure 15.1(*a*) shows the coordinate system selected and 15.1(*b*) shows a typical element subject to the membrane forces.



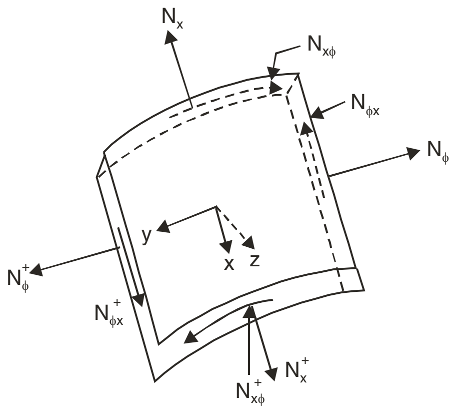


Fig. 15.1

Note:

1. The origin of the coordinate system is at the apex of the shell directrix at mid span.
2. The coordinate x is measured along the crown generator.
3. The coordinate ϕ or y is along the tangent to directrix.
4. The coordinate z is in the radial inward normal.

The membrane forces acting on the element are

1. longitudinal tension N_x per unit width.
2. tension is transverse (ϕ or y) direction N_ϕ per unit width.
3. shear forces $N_{x\phi}$ and $N_{\phi x}$ per unit width.

Measuring ϕ from the crown and taking an element of size $dx \times Rd\phi$ we may note,

$$N_x^+ = N_x + \frac{\partial N_x}{\partial x}dx$$

$$N_\phi^+ = N_\phi + \frac{\partial N_\phi}{\partial \phi}d\phi$$

$$N_{x\phi}^+ = N_{x\phi} + \frac{\partial N_{x\phi}}{\partial x}dx$$

$$N_{\phi x}^+ = N_{\phi x} + \frac{\partial N_{\phi x}}{\partial \phi}d\phi$$

Let X, Y and Z be components of load intensity on the element.
Consider the equilibrium of forces in x-direction.

$$\sum F_x = 0 \rightarrow$$

$$N_x^+Rd\phi - N_xRd\phi + N_{\phi x}^+ dx - N_{\phi x}\, dx + X\, dx\, Rd\phi = 0$$

Substituting for N_x^+ and $N_{\phi x}^+$ we get,

$$\left(N_x + \frac{\partial N_x}{\partial x}dx\right)Rd\phi - N_xRd\phi + \left(N_{\phi x} + \frac{\partial N_{\phi x}}{\partial \phi}d\phi\right)dx - N_{\phi x}dx + XRdxd\phi = 0$$

Simplifying and dividing throughout by $R\, dx\, d\phi$, we get

$$\frac{\partial N_x}{\partial x} + \frac{1}{R}\frac{\partial N_{\phi x}}{\partial \phi} + X = 0 \qquad \text{...eqn. 15.1}$$

Similarly $\sum F_y = 0$, gives

$$\left(N_\phi + \frac{\partial N_\phi}{\partial \phi}d\phi\right)dx - N_\phi dx + \left(N_{x\phi} + \frac{\partial N_{x\phi}}{\partial x}dx\right)Rd\phi - N_{x\phi}Rd_\phi + YRdx\, d\phi$$

Simplifying and dividing throughout by $Rdxd\phi$, we get,

$$\frac{\partial N_{x\phi}}{\partial x} + \frac{1}{R}\frac{\partial N_\phi}{\partial \phi} + Y = 0 \qquad \text{...eqn. 15.2}$$

Consider the equilibrium of forces in z-direction.

i.e. $\sum F_z = 0 \rightarrow$

$$N_\phi^+ dx \sin\left(\frac{dx}{2}\right) + N_\phi dx \sin\frac{d\phi}{2} + ZdxRd\phi = 0.$$

Since, $d\phi$ is small angle, $\sin\left(\frac{d\phi}{2}\right) = \frac{d\phi}{2}$

$$\left(N_\phi + \frac{\partial N_\phi}{\partial \phi} d\phi\right)\frac{d\phi}{2} + N_\phi \frac{d\phi}{2} + Z\,Rdx\,d\phi = 0$$

Neglecting small quantity of higher order, we get,

$$N_\phi + ZR = 0 \qquad \text{...eqn. 15.3}$$

Taking moment about z-axis through the centre of element, we get

$$N_{x\phi} Rd\phi \frac{dx}{2} + N_{x\phi}^+ Rd\phi \frac{dx}{2} - N_{\phi x} dx \frac{Rd\phi}{2} - N_{\phi x}^+ dx \frac{Rd\phi}{2} = 0$$

Substituting for $N_{x\phi}^+$ and $N_{\phi x}^+$ and then neglecting small quantities of higher order, we get,

$$N_{x\phi} = N_{\phi x}.$$

Hence, the equations of equilibrium are

$$\left.\begin{aligned} \frac{\partial N_x}{\partial x} + \frac{\partial N_{x\phi}}{Rd\phi} + X &= 0 \\ \frac{\partial N_{x\phi}}{\partial x} + \frac{\partial N_\phi}{Rd\phi} + Y &= 0 \\ \text{and} \qquad N_\phi + ZR &= 0 \end{aligned}\right\} \qquad \text{...eqn. 15.4}$$

For any given loading the above three equations of equilibrium can be solved to get the membrane forces. N_x, N_ϕ and $N_{x\phi}$.

Example 15.1. Find the expressions for membrane forces in a circular cylindrical shell roof subjected to self weight only.

Solution.

Self weight be g/unit surface area.

Then its X, Y, Z components are given by (Ref. Fig. 15.2)

$$X = 0$$

$$Y = g \sin\phi$$

$$Z = g \cos\phi$$

From third equation of equilibrium, we get,

$$N_\phi = -ZR = -gR \cos\phi$$

From second equation of equilibrium,

$$\frac{\partial N_{x\phi}}{\partial x} + \frac{\partial N_\phi}{R\partial\phi} + Y = 0$$

$$\frac{\partial N_{x\phi}}{\partial x} + \frac{1}{R}(-gR)(-\sin\phi) + g\sin\phi = 0$$

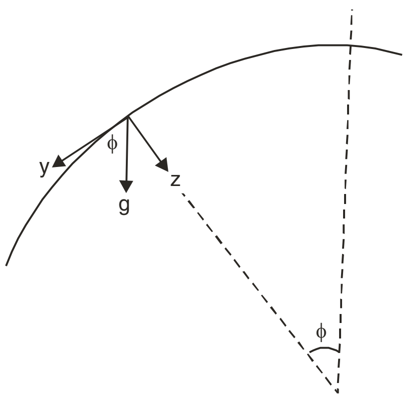


Fig. 15.2

i.e. $$\frac{\partial N_{x\phi}}{\partial x} = -2g \sin\phi$$

Integrating w.r.t. x on both side we get,

$$N_{x\phi} = -2gx \sin\phi + f_1(\phi)$$

where $f_1(\phi)$ is constant of integration.

The boundary condition is,

at $x = 0$, $N_{x\phi} = 0$ due to symmetry.

From this condition, we get,

$$0 = 0 + f_1(\phi).$$

$$\therefore \quad N_{x\phi} = -2gx \sin\phi$$

From equation of equilibrium (1), we have

$$\frac{\partial N_x}{\partial x} + \frac{1}{R}\frac{\partial}{\partial \phi}(-2gx \sin\phi) = 0$$

i.e. $$\frac{\partial N_x}{\partial x} = \frac{1}{R} 2gx \cos\phi$$

$$\therefore \quad N_x = \frac{1}{R} 2g\left(\frac{x^2}{2}\right)\cos\phi + f_2(\phi)$$

$$= \frac{gx^2}{R}\cos\phi + f_2(\phi)$$

where $f_2(\phi)$ is the constant of integration.

Boundary condition: Since the end frame is assumed to be simple support, at $x = L/2$, $N_x = 0$

$$0 = \frac{g}{R}\frac{L^2}{4}\cos\phi + f_2(\phi)$$

or $$f_2(\phi) = -\frac{g}{R}\frac{L^2}{4}\cos\phi$$

Hence, $$N_x = \frac{gx^2}{R}\cos\phi - \frac{g}{R}\frac{L^2}{4}\cos\phi$$

$$= -\frac{gL^2}{4R}\left(1 - \frac{4x^2}{L^2}\right)\cos\phi$$

Thus, the expressions for the membrane forces are

$$\left.\begin{aligned} \mathbf{N_\phi} &= \mathbf{-gR \cos\phi} \\ \mathbf{N_{x\phi}} &= \mathbf{-2gx \sin\phi} \\ \text{and} \quad \mathbf{N_x} &= \mathbf{-\frac{gL^2}{4R}\left(1 - \frac{4x^2}{L^2}\right)\cos\phi} \end{aligned}\right\} \quad \text{...eqn. 15.5}$$

Example 15.2. Determine the expressions for membrane forces in a circular cylindrical shell subject to a snow load of p_0/unit horizontal area.

Solution. Load is p_0/unit horizontal area.

It is equal to $p_0 \cos\phi$ per unit surface (which can be easily seen from Fig. 15.3.)

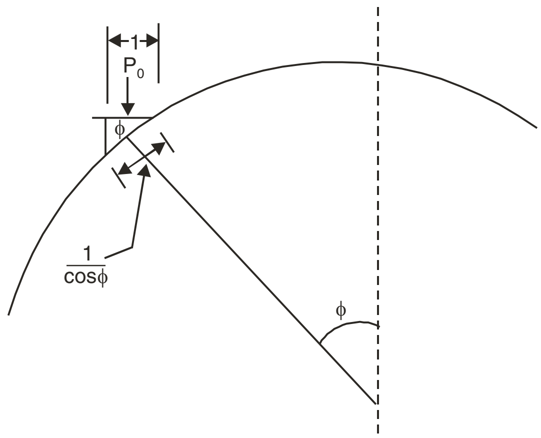


Fig. 15.3

$$\therefore \quad X = 0$$

$$Y = p_0 \cos\phi . \sin\phi$$

$$Z = p_0 \cos\phi . \cos\phi = p_0 \cos^2\phi$$

From equation of equilibrium (3),

$$N_\phi = -ZR = -p_0 \cos^2\phi R$$

$$= -p_0 R \cos^2\phi$$

From equation of equilibrium in ϕ-direction,

$$\frac{\partial N_{x\phi}}{\partial x} + \frac{1}{R}\frac{\partial N_\phi}{\partial \phi} + Y = 0$$

i.e.
$$\frac{\partial N_{x\phi}}{\partial x} + \frac{1}{R}(-p_0 R) 2\cos\phi(-\sin\phi) + p_0 \cos\phi \sin\phi = 0$$

i.e.
$$\frac{\partial N_{x\phi}}{\partial x} = -3p_0 \cos\phi \sin\phi$$

$$\therefore \quad N_{x\phi} = -3p_0 x \cos\phi \sin\phi + F_1(\phi)$$

Due to symmetry, at $x = 0$, $N_{x\phi} = 0$

$$\therefore \quad 0 = 0 + f_1(\phi).$$

Hence,
$$N_{x\phi} = -3p_0 x \cos\phi \sin\phi$$

$$= -\frac{3}{2} p_0 x \sin 2\phi$$

From equation of equilibrium in x-direction,

$$\frac{\partial N_x}{\partial x} + \frac{1}{R}\frac{\partial N_{x\phi}}{\partial \phi} + X = 0.$$

i.e.
$$\frac{\partial N_x}{\partial x} + \frac{1}{R}\left(-\frac{3}{2}p_0\right)x\,2\cos 2\phi = 0$$

or
$$\frac{\partial N_x}{\partial x} = \frac{3p_0 x \cos 2\phi}{R}$$

$\therefore$
$$N_x = \frac{3p_0}{R}\frac{x^2}{2}\cos 2\phi + f_2(\phi)$$

As end frames are treated as simple supports, at $x = L/2$, $N_x = 0$.

$\therefore$
$$0 = \frac{3p_0}{R}\frac{L^2}{8}\cos 2\phi + f_2(\phi)$$

or
$$f_2(\phi) = -\frac{3p_0}{R}\frac{L^2}{8}\cos 2\phi$$

Hence,
$$N_x = \frac{3p_0}{R}\frac{x^2}{2}\cos 2\phi - \frac{3p_0}{R}\frac{L^2}{8}\cos 2\phi$$

$$= -\frac{3p_0}{R}\frac{L^2}{8}\left[1 - \frac{4x^2}{L^2}\right]\cos 2\phi$$

Thus in this case

$$\mathbf{N_\phi = -\,p_0R\,\cos^2\phi}$$

$$\mathbf{N_{k\phi} = -\frac{3}{2}p_0 x \sin 2\phi} \qquad \text{...eqn. 15.6}$$

and
$$\mathbf{N_x = -\frac{p_0L^2}{8R}3\left(1 - \frac{4x^2}{L^2}\right)\cos 2\phi}$$

Example 15.3. Find the membrane forces in a circular cylindrical shell subject to a sinusoidal loading of intensity $\frac{4g}{\pi}\cos\frac{\pi x}{L}$ per unit surface area acting vertically downward.

Solution. Now vertical downward load intensity

$$= \frac{4g}{\pi}\cos\frac{\pi x}{L} \text{ per unit surface area}$$

[**Note:** It is first term of equivalent Fourier series for self weight].

Load components are,

$$X = 0$$

$$Y = \frac{4g}{\pi}\cos\frac{\pi x}{L}\cdot\sin\phi$$

$$Z = \frac{4g}{\pi}\cos\frac{\pi x}{L}\cos\phi$$

From third equation of equilibrium,

$$N_{\phi} = -ZR = -\frac{4gR}{\pi}\cos\frac{\pi x}{L}\cos\phi$$

From the equation of equilibrium in ϕ-direction,

$$= \frac{\partial N_{x\phi}}{\partial x} + \frac{1}{R}\frac{\partial N_{\phi}}{\partial \phi} + Y = 0, \text{ we get,}$$

$$\frac{\partial N_{x\phi}}{\partial x} + \frac{1}{R}\left(-\frac{4gR}{\pi}\right)\cos\frac{\pi x}{L}(-\sin\phi) + \frac{4g}{\pi}\cos\frac{\pi x}{L}\sin\phi = 0.$$

$$\frac{\partial N_{x\phi}}{\partial x} = -\frac{8g}{\pi}\cos\frac{\pi x}{L}\sin\phi$$

$\therefore$ $$N_{x\phi} = -\frac{8g}{\pi}\cdot\frac{L}{\pi}\sin\frac{\pi x}{L}\cdot\sin\phi + f_1(\phi)$$

Due to symmetry at $x = 0$, $N_{x\phi} = 0$

$\therefore$ $0 = 0 + f_1(\phi)$ *i.e.* $f_1(\phi) = 0$

$\therefore$ $$N_{x\phi} = -\frac{8g}{\pi^2}L\sin\frac{\pi x}{L}\cdot\sin\phi$$

From equation of equilibrium in x-direction,

$$\frac{\partial N_x}{\partial x} + \frac{1}{R}\frac{\partial N_{x\phi}}{\partial \phi} + X = 0$$

i.e. $$\frac{\partial N_x}{\partial x} + \frac{1}{R}\left(-\frac{8gL}{\pi^2}\right)\sin\frac{\pi x}{L}\cos\phi = 0$$

$\therefore$ $$\frac{\partial N_x}{\partial x} = \frac{8gL}{\pi^2 R}\sin\frac{\pi x}{L}\cdot\cos\phi$$

$$N_x = \frac{8gL}{\pi^2 R}\times\frac{L}{\pi}\left(-\cos\frac{\pi x}{L}\right)\cos\phi + f_2\phi$$

$$= -\frac{8gL^2}{\pi^3 R}\cos\frac{\pi x}{L}\cdot\cos\phi + f_2(\phi)$$

At $$x = \frac{L}{2},\ N_x = 0.$$

$\therefore$ $$0 = 0 + f_2(\phi)$$

$\therefore$ $$N_x = -\frac{8gL^2}{\pi^3 R}\cos\frac{\pi x}{L}\cdot\cos\phi$$

Thus in this case,

$$\mathbf{N_\phi = -\frac{4gR}{\pi}\cos\frac{\pi x}{L}\cos\phi}$$

$$\mathbf{N_{x\phi} = -\frac{8gL}{\pi^2}\sin\frac{\pi x}{L}\cdot\sin\phi} \qquad \text{...eqn. 15.7}$$

and $$\mathbf{N_x = -\frac{8gL^2}{\pi^3 R}\cos\frac{\pi x}{L}\cdot\cos\phi}$$

15.2 CYLINDRICAL SHELLS WITH PARABOLIC, CATENARY AND CYCLOID DIRECTRICES

Intrinsic equation is that equation which relates radius of curvature and the angle. It is interesting to note that the intrinsic equations for parabolic, catenary, circular and cycloid are having general form.

$$R = R_0 \cos^n\phi$$

where R_o—radius of curvature at crown (ϕ = 0)

If n = –3, it is parabolic
= –2, it is catenary
= 0, it is circular, and
= 1, it is cycloid.

The shapes of these curves look as shown in Fig. 15.4.

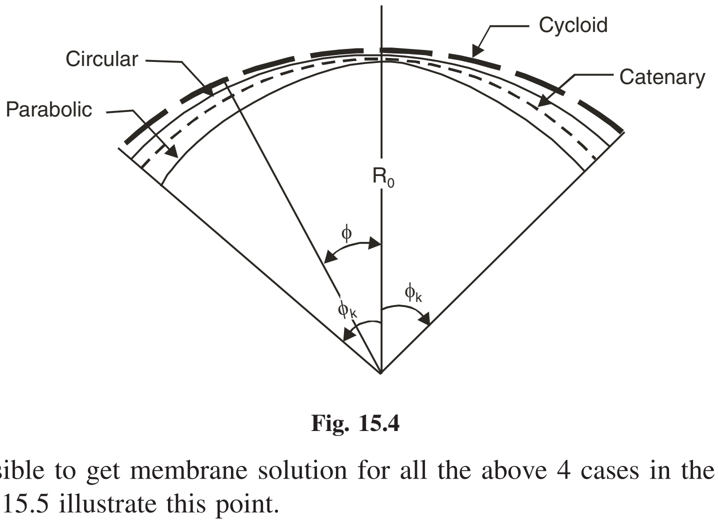


Fig. 15.4

Hence, it is possible to get membrane solution for all the above 4 cases in the general form. The examples 15.4 and 15.5 illustrate this point.

Example 15.4 Give membrane analysis of cylindrical shells with parabolic, catenary, circular and cycloid subject to self weight *g*/unit surface area.

Solution. For self weight,

$$X = 0$$
$$Y = g \sin\phi$$
$$Z = g \cos\phi$$

From equation of equilibrium of forces in *z*-direction,

$$N_\phi = -ZR$$
$$= -g \cos\phi . R_0 \cos^n\phi$$
$$= -gR_0 \cos^{n+1}\phi$$

From equation of equilibrium of forces in ϕ-direction, we have

$$\frac{\partial N_{x\phi}}{\partial x} + \frac{1}{R}\frac{\partial N_\phi}{\partial \phi} + Y = 0$$

$$\frac{\partial N_{x\phi}}{\partial x} + \frac{1}{R}(-gR_0)(n+1)\cos^n\phi(-\sin\phi) + g\sin\phi = 0$$

$$\frac{\partial N_{x\phi}}{\partial x} + \frac{(n+1)gR_0}{R_0\cos^n\phi}\cos^n\phi\sin\phi + g\sin\phi = 0$$

i.e.
$$\frac{\partial N_{x\phi}}{\partial x} + (n+2)\sin\phi = 0$$

i.e.
$$\frac{\partial N_{x\phi}}{\partial x} = -g(n+2)\sin\phi$$

Integrating both sides w.r.t. *x*, we get,

$$N_{x\phi} = -gx(n+2)\sin\phi + f_1(\phi)$$

where $f_1(\phi)$ is constant of integration.

The boundary condition is, due to symmetry, at $x = 0$, $N_{x\phi} = 0$.

i.e. $0 = 0 + f_1(\phi)$ *i.e.* $f_1(\phi) = 0$.

Hence, $N_{x\phi} = -gx\,(n+2)\sin\phi$

From the equation of equilibrium of forces in *x*-direction, we have,

$$\frac{\partial N_x}{\partial x} + \frac{1}{R}\frac{\partial N_{x\phi}}{\partial \phi} + X = 0.$$

i.e.
$$\frac{\partial N_x}{\partial x} + \frac{1}{R_0\cos^n\phi}\cdot(-gx)(n+2)\cos\phi = 0$$

i.e.
$$\frac{\partial N_x}{\partial x} = \frac{gx(n+2)}{R_0\cos^{n-1}\phi}$$

$\therefore$
$$N_x = \frac{gx^2(n+2)}{2R_0\cos^{n-1}\phi} + f_2(\phi)$$

As end frames are assumed as simple supports,

at $$x = \frac{L}{2},\ N_x = 0.$$

i.e. $$0 = \frac{gL^2(n+2)}{8R_0 \cos^{n-1}\phi} + f_2(\phi)$$

or $$f_2(\phi) = -\frac{gL^2}{8R_0}\frac{(n+2)}{\cos^{n-1}\phi}$$

$\therefore$ $$N_x = \frac{gx^2(n+2)}{2R_0\cos^{n-1}\phi} - \frac{gL^2}{8R_0}\frac{(n+2)}{\cos^{n-1}\phi}$$

$$= -\frac{gL^2}{8R_0}\frac{(n+2)}{\cos^{n-1}\phi}\left(1 - \frac{4x^2}{L^2}\right)$$

Thus,

$$\mathbf{N_\phi = -gR_0\cos^{n+1}\phi}$$

$$\mathbf{N_{x\phi} = -gx(n+2)\sin\phi} \qquad \text{...eqn. 15.8}$$

$$\mathbf{N_x = -\frac{gL^2}{8R_0}\frac{(n+2)}{\cos^{n-1}\phi}\left(1 - \frac{4x^2}{L^2}\right)}$$

$n = -3$ is for parabolic, $n = -2$ is for catenary, $n = 0$ is for circular and $n = 1$ is for cycloid.

Example 15.5. Find the membrane forces in the cylindrical shells with parabolic, catenary, circular and cycloid directrices due to snow load of intensity p_0 per unit horizontal area.

Solution. Let the intrinsic equation be $R = R_0 \cos^n\phi$

Snow load of p_0/per unit horizontal area

$= p_0 \cos\phi$/per unit surface area

$\therefore$ Load components are

$$X = 0$$
$$Y = p_0 \cos\phi \sin\phi$$
$$Z = p_0 \cos^2\phi$$

From the equation of equilibrium of forces in z-direction, we get,

$$N_\phi = -ZR$$

$$= -p_0\cos^2\phi \cdot R_0\cos^n\phi$$

$$= -p_0 R_0 \cos^{n+2}\phi$$

From the equation of equilibrium of forces in ϕ-direction, we get,

$$\frac{\partial N_{x\phi}}{\partial x} + \frac{1}{R}\frac{\partial N_\phi}{\partial \phi} + Y = 0$$

$$\frac{\partial N_{x\phi}}{\partial x}+\frac{1}{R_0\cos^n\phi}\cdot(-p_0R_0)(n+2)\cos^{n+1}\phi(-\sin\phi)+p_0\cos\phi\sin\phi=0$$

i.e.
$$\frac{\partial N_{x\phi}}{\partial x}+p_0(n+2)\sin\phi\cos\phi+p_0\cos\phi\sin\phi=0$$

i.e.
$$\frac{\partial N_{x\phi}}{\partial x}=-p_0(n+3)\sin\phi\cos\phi$$

$$=-p_0\frac{(n+3)}{2}\sin 2\phi$$

∴
$$N_{x\phi}=-\frac{p_0x(n+3)}{2}\times\sin 2\phi+f_1(\phi).$$

From boundary condition, $N_{x\phi}=0$ at $x=0$, we get,

$$0=0+f_1(\phi)\text{ or }f_1(\phi)=0$$

∴
$$N_{x\phi}=-\frac{p_0x(n+3)}{2}\sin 2\phi$$

From the equation of equilibrium of forces in x-direction, we have

$$\frac{\partial N_x}{\partial x}+\frac{1}{R}\frac{\partial N_\phi}{\partial\phi}+X=0$$

i.e.
$$\frac{\partial N_x}{\partial x}+\frac{1}{R_0\cos^n\phi}(-p_0x)\frac{n+3}{2}2\cos 2\phi=0$$

$$\frac{\partial N_x}{\partial x}=p_0x(n+3)\frac{\cos 2\phi}{R_0\cos^n\phi}$$

∴
$$N_x=p_0\frac{x^2}{2}(n+3)\frac{\cos 2\phi}{R_0\cos^n\phi}+f_2(\phi)$$

From the boundary condition, $N_x=0$ at $x=L/2$, we get,

$$0=p_0\frac{L^2}{8}(n+3)\frac{\cos 2\phi}{R_0\cos^n\phi}+f_2(\phi).$$

$$f_2(\phi)=-p_0\frac{L^2}{8}(n+3)\frac{\cos 2\phi}{R_0\cos^n\phi}$$

Hence,
$$N_x=\frac{p_0x^2}{2}(n+3)\frac{\cos 2\phi}{R_0\cos^n\phi}-\frac{P_0L^2}{8}(n+3)\frac{\cos 2\phi}{R_0\cos^n\phi}$$

$$= -\frac{p_0 L^2}{8}(n+3)\frac{\cos 2\phi}{R_0 \cos^n \phi}\left(1-\frac{4x^2}{L^2}\right)$$

Thus, the solution is

$$\mathbf{N_\phi = -p_0 R_0 \cos^{n+2}\phi}$$

$$\mathbf{N_{x\phi} = \frac{-p_0 x(n+3)}{2}\sin 2\phi} \qquad \text{… eqn. 15.9}$$

$$\mathbf{N_x = -\frac{p_0 L^2}{8}(n+3)\frac{\cos 2\phi}{R_0 \cos^n \phi}\left(1-\frac{4x^2}{L^2}\right)}$$

$n = -3$ is for parabolic
$n = -2$ is for catenary
$n = 0$ is for circular and
$n = 1$ is for cycloid

The membrane forces for cylindrical shells with different directrices due to self weight and snow load are shown in Table 15.1.

Table 15.1 Membrane forces

Directrix	N_ϕ		$N_{x\phi}$		N_x	
	DL	*SL*	*DL*	*SL*	*DL*	*SL*
Parabolic $n = -3$	$-\frac{1}{\cos^2\phi}$	$-\frac{1}{\cos\phi}$	$\sin\phi$	0	$\cos^4\phi$	0
Catenary $n = -2$	$-\frac{1}{\cos\phi}$	-1	0	$-\frac{\sin 2\phi}{2}$	0	$-\cos 2\phi \cos^2\phi$
Circular $n = 0$	$-\cos\phi$	$-\cos^2\phi$	$-2\sin\phi$	$-\frac{3}{2}\sin 2\phi$	$-2\cos\phi$	$-3\cos 2\phi$
Cycloid $n = 1$	$-\cos^2\phi$	$-\cos^3\phi$	$-3\sin\phi$	$-2\sin 2\phi$	-3	$-\frac{4\cos 2\phi}{\cos\phi}$
Common multiplying factor	gR_0	p_0R_0	gx	p_0x	$\frac{gL^2}{8R_0}\left(1-\frac{4x^2}{L^2}\right)$	$\frac{p_0L^2}{8R_0}\left(1-\frac{4x^2}{L^2}\right)$

15.3 CYLINDRICAL SHELL WITH SEMI-ELLIPTIC DIRECTRIX

Figure 15.5 shows a semiellipse with major axis = $2a$ and minor axis = $2b$. Its radius of curvature at any central angle ϕ is given by

$$R = \frac{a^2 b^2}{\left(a^2 \sin^2\phi + b^2\cos^2\phi\right)^{3/2}}$$

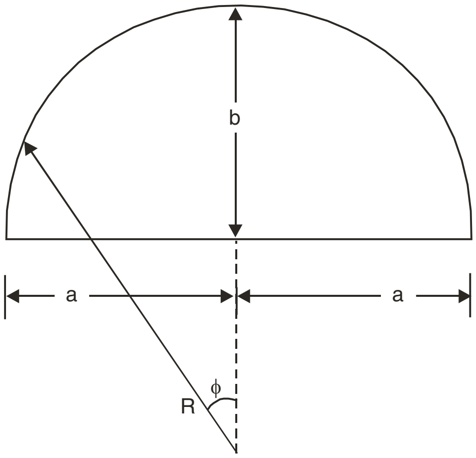


Fig. 15.5

(1) **For self weight** $X = 0$, $Y = g\sin\phi$, $Z = g\cos\phi$

From equation of equilibrium for forces in z-direction, we have

$$N_\phi = -ZR$$

$$= -g\frac{a^2 b^2\cos\phi}{\left(a^2\sin^2\phi + b^2\cos^2\phi\right)^{3/2}}$$

From equation of equilibrium in ϕ-direction,

$$\frac{\partial N_{x\phi}}{\partial x} + \frac{1}{R}\frac{\partial N_\phi}{\partial \phi} + Y = 0$$

$$\therefore \qquad \frac{\partial N_{x\phi}}{\partial x} = -\frac{\left(a^2\sin^2\phi + b^2\cos^2\phi\right)^{3/2}}{a^2b^2} g a^2 b^2 \left[\frac{-\sin\phi}{\left(a^2\sin^2\phi + b^2\cos^2\phi\right)^{3/2}}\right.$$

$$\left. + \cos\phi\frac{3}{2}\left(a^2\sin^2\phi + b^2\cos^2\phi\right)^{-5/2}\left(2a^2\sin\phi\cos\phi + 2b^2\cos\phi(-\sin\phi)\right)\right]$$

$$= -g\sin\phi\left[1 + \frac{3\left(a^2\cos^2\phi - b^2\cos^2\phi\right)}{\left(a^2\sin^2\phi + b^2\cos^2\phi\right)} + 1\right]$$

$$\therefore \qquad N_{x\phi} = -gx\sin\phi\left[2 + \frac{3\left(a^2 - b^2\right)\cos^2\phi}{a^2\sin^2\phi + b^2\cos^2\phi}\right] + f_1(\phi)$$

[**Note:** In differentiating ϕ, $u = \cos\phi$ and $v = \left(a^2\sin^2\phi + b^2\cos^2\phi\right)^{-3/2}$. Hence, $\frac{d}{dx}(uv)$]

At $x = 0$, $N_{x\phi} = 0$ (due to symmetry)

$\therefore$ $f_1(\phi) = 0$

$$\therefore \qquad N_{x\phi} = -gx\sin\phi\left[2 + \frac{3\left(a^2 - b^2\right)\cos^2\phi}{a^2\sin^2\phi + b^2\cos^2\phi}\right]$$

From equation of equilibrium of forces in x-direction, we have

$$\frac{\partial N_x}{\partial x} + \frac{1}{R}\frac{\partial N_{x\phi}}{\partial \phi} + X = 0.$$

$$\frac{\partial N_x}{\partial x} = -\frac{1}{R}\frac{\partial N_{x\phi}}{\partial \phi} \qquad \text{since } X = 0\cdot$$

$$= +\frac{\left(a^2\sin^2\phi + b^2\cos^2\phi\right)^{3/2}}{a^2b^2}gx\left[2\cos\phi + 3\left(a^2 - b^2\right)\left\{\frac{\cos^3\phi - 2\cos\phi\sin^2\phi}{a^2\sin^2\phi + b^2\cos^2\phi}\right\}\right.$$

$$\left. - \frac{\cos^2\phi\sin\phi\, 2\left(a^2 - b^2\right)\cos\phi\sin\phi}{a^2\sin^2\phi + b^2\cos^2\phi}\right]$$

$$= gx\cos\phi\left[\frac{2ab}{\alpha^3} + \frac{3\left(a^2 - b^2\right)}{ab\alpha}\left\{\cos^2\phi - \frac{2\alpha^2}{b^2}\sin^2\phi\right\}\right]$$

where

$$\alpha = \frac{ab}{\left(a^2\sin^2\phi + b^2\cos^2\phi\right)^{3/2}}$$

$$\therefore \qquad N_x = \frac{gx^2}{2}\cos\phi\left[\frac{2ab}{\alpha^3} + \frac{3\left(a^2 - b^2\right)}{ab\alpha}\left\{\cos^2\phi - \frac{2\alpha^2}{b^2}\sin^2\phi\right\}\right] + f_2(\phi)$$

From B-C, $N_x = 0$ at $x = L/2$, $f_2(\phi)$ may be found and on simplification we get,

$$N_x = \frac{-gL^2}{8}\left(1 - \frac{4x^2}{L^2}\right)\cos\phi\left[\frac{2ab}{\alpha^3} + \frac{3\left(a^2 - b^2\right)}{ab\alpha}\right]\left[\cos^2\phi - \frac{2\alpha^2}{b^2}\sin^2\phi\right]$$

(2) **For snow load:**

Let snow load be p_0 per unit horizontal load. Then

$$X = 0, \quad Y = p_0\cos\phi\sin\phi \quad Z = p_0\cos^2\phi$$

Proceeding on the same line as earlier, it can be shown that

$$N_\phi = -p_0 \frac{a^2 b^2 \cos^2 \phi}{a^2 \sin^2 \phi + b^2 \cos^2 \phi}$$

$$N_{x\phi} = -3 p_0 x \frac{a^2 \cos\phi \sin\phi}{a^2 \sin^2 \phi + b^2 \cos^2 \phi}$$

$$N_x = -\frac{3 p_0 L^2}{8}\left(1 - \frac{4x^2}{L^2}\right)\frac{b^2 \cos^2 \phi - a^2 \sin^2 \phi}{\left(a^2 \sin^2 \phi + b^2 \cos^2 \phi\right)^{1/2}}$$

15.4 COMMENTS ON MEMBRANE THEORY

1. A thin shell acts partially as an arch and partially as a beam. Due to arch action (in the form of N_ϕ) the load is transferred to edge beam and due to beam action (in the form of $N_{x\phi}$) the load is transferred to the end frame.
2. If the directrix choosen is the funicular curve of the applied loading (catenary for self weight and parabola for snow load), the shell degenerates itself into a series of independent arches and beam action completely disappears.
3. The distribution of longitudinal force N_x across the cross section is not linear as in an ordinary beam, except in the case of circular directrix under self weight.
4. The variation of N_ϕ is independent of x.
5. For a shell with free edges, N_ϕ and $N_{x\phi}$ should be zero at edges. But membrane theory gives definite values. The membrane theory fails to give good results at edges.

In actual shells, edges bend in transverse direction (to avoid these unbalanced forces) introducing transverse moment M_ϕ and radial shear. Earlier research was to assess these edge perturbations and superpose them with membrane values.

QUESTIONS

1. Derive the differential equations of equilibrium using membrane forces only. Find the stresses N_x, N_y and N_{xy} due to uniformly distributed load acting on the shell surface.
2. Find the membrane stresses in a circular cylindrical shell subject to snow load.
3. Find the membrane forces in a circular cylindrical shell subject to a sinusoidal load.
4. What is intrinsic equation? Give it for
   (*a*) Circular
   (*b*) Conoid
   (*c*) Parabolic and
   (*d*) Catenary directrices.
   Derive the general membrane solution for above type of cylindrical shells subjected to self weight only.
5. Comment on membrane theory for cylindrical shells.

Chapter

16

Bending Theory of Cylindrical Shell Roofs

In the previous chapter, we have seen that membrane theory do not satisfy the edge conditions. This was realized as early as in 1930. Though considerable load transfer in shells is by membrane action, bending is not completely avoided. The moments and transverse shears act on the shell element. Hence, in deriving the equilibrium equations for elements, moments and transverse shears should be considered. Finsterwalder and Dishinger of Germany gave a bending theory in 1930. Schorer of America gave a simplified solution in 1936. The contributions of Donnel (1933–34), Karman (1941) and Jenkins (1947) lead to the bending theory known as DKJ theory which can be applied to circular cylindrical shells of all dimensions. In this chapter, shell element and forces acting on it are explained, making sign conventions clear. Then equations of equilibrium are derived and relations between stress resultants and radial displacement '*w*' are derived. As it is highly impossible to go ahead with exact relations to find the solution, the assumptions made by various researchers are presented and the solution by Schorer and DKJ are presented. Statical checks to be applied are also presented.

16.1 A TYPICAL SHELL ELEMENT

A typical shell element and various forces acting on it are shown in Fig. 16.1.

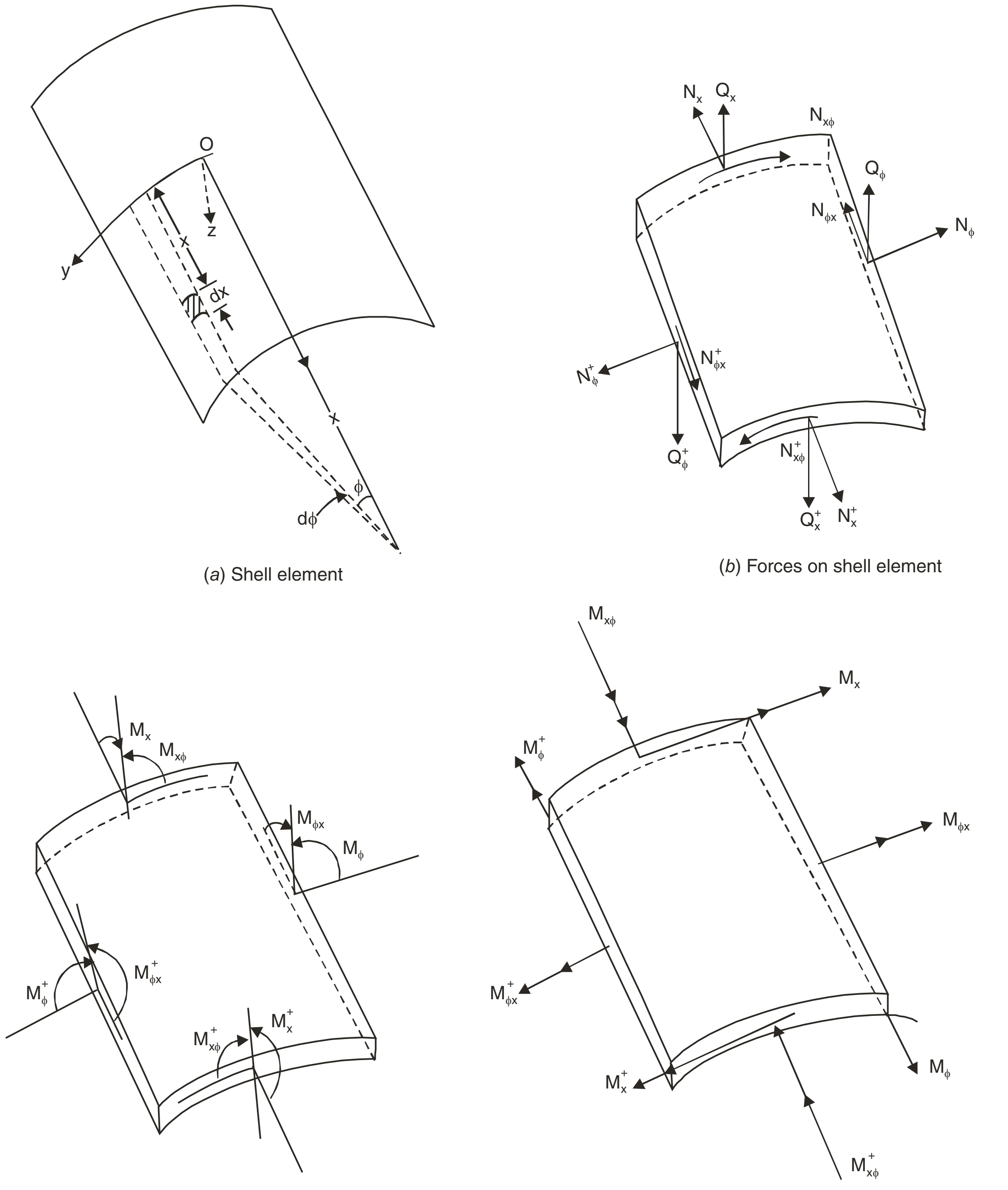
The coordinate system x, y, z is same as the one selected in membrane theory.

In the figure, all forces and moments are shown in their positive senses. It may be noted that the sign convention followed is that on positive face if the force is acting in positive direction, it is positive. At the same time, if the force is acting in negative direction on negative face, then also it is positive force. The moments are positive if they are produced by positive forces acting in positive direction of z. This convention results into tensile forces and sagging moments as positive. For shears and twisting moment one should carefully note the sign conventions used. It is obvious that in the element shown:

$$N_x^+ = N_x + \frac{\partial N_x}{\partial x} dx \qquad M_x^+ = M_x + \frac{\partial M_x}{\partial x} dx$$

$$N_\phi^+ = N_\phi + \frac{\partial N_\phi}{\partial \phi} d\phi \qquad M_\phi^+ = M_\phi + \frac{\partial M_\phi}{\partial \phi} d\phi$$

$$N_{x\phi}^+ = N_{x\phi} + \frac{\partial N_{x\phi}}{\partial x} dx \qquad M_{x\phi}^+ = M_{x\phi} + \frac{\partial M_{x\phi}}{\partial x} dx$$



(c) Moments on shell elements

(d) Another method of representing moments (4 closed right hand fingers indicate direction of moment and stretched thumb indicates axis about which moment act)

Fig. 16.1

$$N_{\phi x}^{+} = N_{\phi x} + \frac{\partial N_{\phi x}}{\partial \phi} d\phi$$

$$M_{\phi x}^{+} = M_{\phi x} + \frac{\partial M_{\phi x}}{\partial \phi} d\phi$$

$$Q_x^{+} = Q_x + \frac{\partial Q_x}{\partial x} dx$$

X, Y, Z are the load components in x, y, z directions respectively.

$$Q_\phi^{+} = Q_\phi + \frac{\partial Q_\phi}{\partial \phi} d\phi$$

16.2 EQUATIONS OF EQUILIBRIUM

Three equations of equilibrium can be found by considering force equilibrium in x, y and z directions and another three by considering moment equilibrium. It may be noted that all forces and moments shown on the element are in per unit length.

Σ Forces in x-direction = 0 $\rightarrow$

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{x\phi}}{R \partial \phi} + X = 0 \qquad \text{...eqn. 16.1}$$

Σ Forces in y-direction = 0 $\rightarrow$

$$\left(N_\phi + \frac{\partial N_\phi}{\partial \phi} d\phi\right) dx - N_\phi dx + \left(N_{x\phi} + \frac{\partial N_{x\phi}}{\partial x} dx\right) R d\phi$$

$$-N_{x\phi} R d\phi - \left[Q_\phi dx \frac{d\phi}{2} + \left(Q_\phi + \frac{\partial Q_\phi}{\partial \phi} d\phi\right) dx \frac{d\phi}{2}\right] + Y dx R d\phi = 0.$$

Neglecting small quantities of higher order and dividing throughout by $Rdxd\phi$, we get,

$$\frac{\partial N_{x\phi}}{\partial x} + \frac{\partial N_\phi}{R \partial \phi} - \frac{Q_\phi}{R} + Y = 0 \qquad \text{...eqn. 16..2}$$

Σ Forces in z-direction = 0 $\rightarrow$

$$\left(Q_x + \frac{\partial Q_x}{\partial x} dx\right) R\, d\phi - Q_x R d\phi + \left(Q_\phi + \frac{\partial Q_\phi}{\partial \phi} d\phi\right) dx$$

$$-Q_x dx + \left(N_\phi + \frac{\partial N_\phi}{R \partial \phi} d\phi\right) dx \frac{d\phi}{2} + N_\phi dx \frac{d\phi}{2} + Z dx\, R d\phi = 0$$

Neglecting small quantities of higher order and dividing throughout by $Rdxd\phi$, we get,

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_\phi}{R \partial \phi} + \frac{N_\phi}{R} + Z = 0 \qquad \text{...eqn. 16.3}$$

Now consider the moment equilibrium conditions.

Σ Moments in x-direction = 0 $\rightarrow$

$$\left(M_x + \frac{\partial M_x}{\partial x}dx\right)R\,d\phi - M_x R\,d\phi + \left(M_{\phi x} + \frac{\partial M_{\phi x}}{\partial \phi}d\phi\right)dx - M_{\phi x}dx - Q_x R d\phi\, dx = 0$$

Simplifying we get,

$$\frac{\partial M_x}{\partial x} + \frac{1}{R}\frac{\partial M_{x\phi}}{\partial \phi} - Q_x = 0 \qquad \text{...eqn. 16.4}$$

Σ Moments in y-direction = 0 $\rightarrow$

we get,

$$\frac{\partial M_{x\phi}}{\partial x} + \frac{1}{R}\frac{\partial M_{\phi}}{\partial \phi} - Q_{\phi} = 0 \qquad \text{...eqn. 16.5}$$

Σ Moments about z-axis = 0, gives

$$\left(M_{\phi x} + \frac{\partial M_{\phi x}}{\partial \phi}d\phi\right)dx\frac{d\phi}{2} + M_{\phi x}dx\frac{d\phi}{2} + N_{x\phi}Rd\phi\, dx - N_{\phi x}dx\, Rd\phi = 0$$

Neglecting small quantities of higher order and dividing throughout by $Rdxd\phi$, we get

$$\frac{M_{\phi x}}{R} + N_{x\phi} - N_{\phi x} = 0 \qquad \text{...eqn. 16.6}$$

Thus we have got six equations of equilibrium and 10 unknowns *i.e.* N_x, N_ϕ, $N_{x\phi}$, $N_{\phi x}$, Q_x, Q_ϕ, M_x, M_ϕ, $M_{x\phi}$ and $M_{\phi x}$. Hence, the problem is not statically determinate. After studying deformations all stress resultants are expressed in terms of single displacement *i.e.* 'w' in z-direction. Then a shell equation is derived in terms of 'w' only, which will be 8th order differential equation. Solution of that equation satisfying the boundary conditions gives expression for 'w', using which any stress resultant can be found.

16.3 DEFORMATION IN SHELL

Let the origin of the coordinate system be at the crown of mid-span section. Let

x – be in the longitudinal direction
y or ϕ – be in the tangential direction and
z – be in the radial inward direction.

Consider the point A with coordinates x, ϕ and z, all having positive values. Let u_A—displacement in longitudinal direction, positive in the direction of increasing x.

v_A— displacement along a circle of radius $(R - z)$, positive in the direction of increasing ϕ and
w_A— radial displacement, positive in the inward direction.

Let u, v and w be the displacement of the middle surface of the point which has also the coordinates x and ϕ, but $z = 0$. The relations among these displacements will be derived first, considering the shell is thin. The following additional assumptions are also made:

1. All points lying on a normal to the middle surface before deformation, do the same after deformation also. In other words, shear deformations are considered negligible.
2. The stresses in radial direction (σ_z) is considered negligible and hence, the deformation in z-direction is negligible. In other word point A remains at distance z even after deformation.
3. All displacements are small *i.e.* they are negligible compared to the radius of curvature of the middle surface and that their first derivatives *i.e.* slopes are negligible compared with unity.

In short, we are considering small deflection theory of thin shells.

(*i*) **Relation between u_A and u**

A_0 and A are the original positions of two points (Ref. Fig. 16.2). A_0 is on the middle surface and A is at a distance z from A_0 on the normal to the surface. After deformation let the points move to the positions A_0' and A' respectively. According to the assumptions even now A' is on normal to the middle surface and is at a distance z. From the figure, it is easily seen that,

$$u_A = u - z\frac{\partial w}{\partial x} \qquad \text{...eqn. 16.7}$$

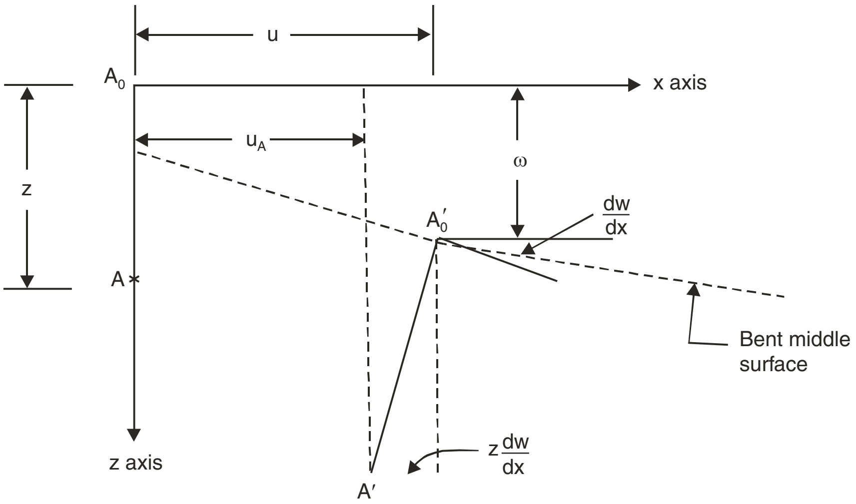


Fig. 16.2

(*ii*) **Relation between v_A and v**

Figure 16.3 shows a transverse section through the shell. The point A_0 on the middle surface is displaced by v along the middle surface to point A_0'. The point A which was at distance z from A_0 is displaced to A' and it remains at right angles to middle surface at A_0'.

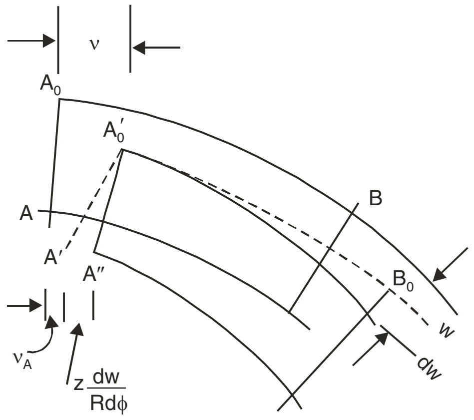


Fig. 16.3

Let A'' be at right angles to middle surface at A_0' at a distance z. Then the displacement of A may be looked as consists of two parts:

(*i*) A moves to A'' due to shortening of radius $= v\dfrac{R-z}{R}$

(*ii*) Then A'' moves to A' due to rotation by angle $\dfrac{\partial w}{R\partial\phi}$ *i.e.* by a distance $z\dfrac{\partial w}{Rd\phi}$

$$\therefore \qquad v_A = \frac{R-z}{R}v - Z\frac{1}{R}\frac{\partial w}{\partial\phi} \qquad \text{...eqn. 16.8}$$

(*iii*) **Relation between w_A and w**

Since, the shell is assumed thin and displacements are considered small, we get,

$$w_A = w. \qquad \text{...eqn. 16.9}$$

16.4 STRAIN DISPLACEMENT RELATIONS

As a next step in assembling shell equation, we establish strain-displacement relations.

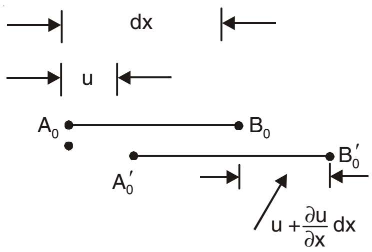


Fig. 16.4

Consider an element AB of length dx at the middle surface. Referring to Fig. 16.4, after deformation let the element A_0B_0 move to the position $A_0'B_0'$. Let A_0 move by distance u in x-direction. Then B_0 moves by some other distance. If the rate of change of this displacement is $\dfrac{\partial u}{\partial x}$, then BB' is equal to $u + \dfrac{\partial u}{\partial x}dx.$

$\therefore$ Strain ε_{xc} in x-direction is given by

$$\varepsilon_{xc} = \frac{\text{Extension}}{\text{Original Length}} = \frac{u + \frac{\partial u}{\partial x}dx - u}{dx}$$

$$= \frac{\partial u}{\partial x}$$

$\therefore$ At A,

$$\varepsilon_x = \frac{\partial u_A}{\partial x} \qquad \text{...eqn. 16.10}$$

Referring to Fig. 16.5, the strain at middle surface in ϕ-direction is given by,

$$\varepsilon_{\phi_c} = \frac{v + \frac{\partial v}{\partial \phi} d\phi - v}{R d\phi} - \frac{w}{R}$$

$$= \frac{1}{R}\left(\frac{\partial v}{\partial \phi} - w\right)$$

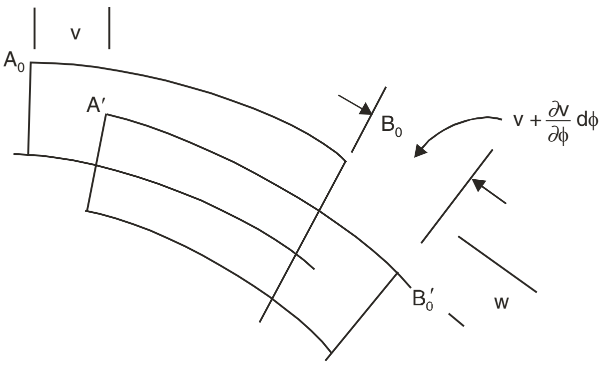


Fig. 16.5

[**Note:** $\frac{w}{R}$ is shortening due to radial inward displacement w].

$\therefore$ At A,

$$\varepsilon_{\phi} = \frac{1}{R - z}\left(\frac{\partial v_A}{\partial \phi} - w_A\right) \quad \text{...eqn. 16.11}$$

Referring to Fig. 16.6, in which shear deformations are shown in their positive senses, we get,

$$\gamma_{x\phi c} = \frac{\partial v}{\partial x} + \frac{1}{R}\frac{\partial v}{\partial \phi}$$

$\therefore$ At A,

$$\gamma_{x\phi} = \frac{\partial v_A}{\partial x} + \frac{1}{R - z}\frac{\partial u_A}{\partial \phi} \quad \text{...eqn. 16.12}$$

From eqns. 16.7 and 16.10, we get,

$$\varepsilon_x = \frac{\partial u_A}{\partial x} = \frac{\partial v}{\partial x} - Z\frac{\partial^2 w}{\partial x^2} \quad \text{...eqn. 16.13}$$

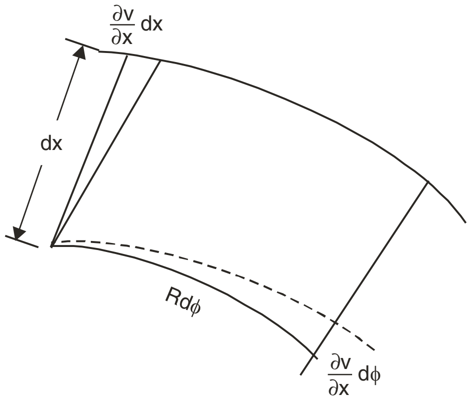


Fig. 16.6

From eqns. 16.8 and 16.11, we get

$$\varepsilon_{\phi} = \frac{1}{R}\frac{\partial v}{\partial \phi} - \frac{Z}{R(R - z)}\frac{\partial^2 w}{\partial \phi^2} - \frac{w}{R - Z}$$

$$= \frac{1}{R}\frac{\partial v}{\partial \phi} - \frac{Z}{R^2\left(1 - \frac{Z}{R}\right)}\frac{\partial^2 w}{\partial \phi^2} - \frac{w}{R\left(1 - \frac{Z}{R}\right)} \quad \text{...eqn. 16.14}$$

Similarly from eqns. 16.9 and 16.12, we get

$$\gamma_{x\phi} = \frac{1}{R - Z}\frac{\partial u}{\partial \phi} + \frac{R - Z}{R}\frac{\partial v}{\partial x} - \left(\frac{Z}{R} + \frac{Z}{R - Z}\right)\frac{\partial^2 w}{\partial x \partial \phi}$$

$$= \frac{1}{R\left(1 - \frac{Z}{R}\right)}\frac{\partial u}{\partial \phi} + \left(1 - \frac{Z}{R}\right)\frac{\partial v}{\partial \phi} - \frac{1}{R}\left(1 + \frac{1}{1 - \frac{Z}{R}}\right)\frac{\partial^2 w}{\partial \phi^2} \quad \text{...eqn. 16.15}$$

16.5 RELATIONS BETWEEN STRESS RESULTANTS AND DISPLACEMENTS

For stress analysis linear elastic theory is used. Hence, from Hooke's law,

$$\sigma_x = \frac{E}{1-\mu^2}\left(\varepsilon_x + \mu\varepsilon_\phi\right) \quad \text{...eqn. 16.16}$$

$$\sigma_\phi = \frac{E}{1-\mu^2}\left(\mu\varepsilon_x + \varepsilon_\phi\right) \quad \text{...eqn. 16.17}$$

and

$$\tau_{xy} = \frac{E}{2(1+\mu)}\gamma_{x\phi} \quad \text{...eqn. 16.18}$$

In the equations 16.16 to 16.18, by substituting the relations of strains with displacements (eqns. 16.13 to 16.15) we get stress-displacement relations.

Now consider an element of unit dimension at middle section as shown in Fig. 16.7. At any distance z from the middle surface the length of element in x-direction is unity but in ϕ-direction, it is different.

Referring to the Fig. 16.6,

$$Rd\phi = 1$$

or

$$d\phi = \frac{1}{R}$$

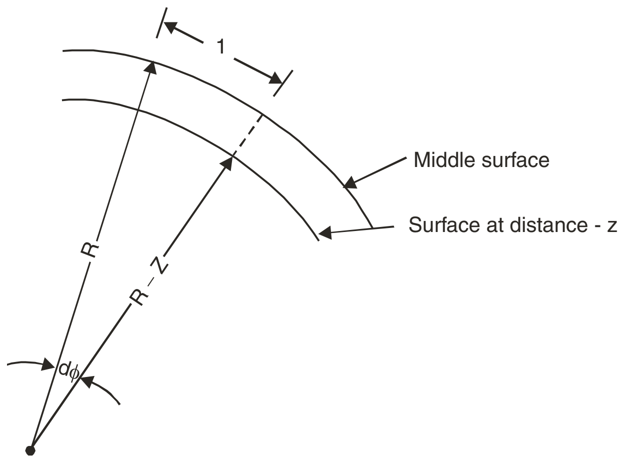


Fig. 16.7

Elemental length in ϕ-direction at distance z from middle surface

$$= (R-Z)d\phi$$

$$= (R-Z)\frac{1}{R}$$

$$= \left(1-\frac{Z}{R}\right)$$

Hence, the stress resultants per unit length are as shown below:

$$N_x = \int_{-h/2}^{h/2} \sigma_x \left(1 - \frac{Z}{R}\right) dz \qquad N_\phi = \int_{-h/2}^{h/2} \sigma_\phi \times 1 \times dz$$

$$N_{x\phi} = \int_{-h/2}^{h/2} \tau_{x\phi} \left(1 - \frac{Z}{R}\right) dz \qquad N_{\phi x} = \int_{-h/2}^{h/2} \tau_{\phi x} \times 1 \times dz$$

$$M_x = \int_{-h/2}^{h/2} \sigma_x \left(1 - \frac{Z}{R}\right) z\,dz \qquad M_\phi = \int_{-h/2}^{h/2} \sigma_\phi \times 1 \times z \times dz$$

$$M_{x\phi} = \int_{-h/2}^{h/2} \tau_{x\phi} \left(1 - \frac{Z}{R}\right) z\,dz \qquad M_{\phi x} = \int_{-h/2}^{h/2} \tau_{\phi x} \times 1 \times z \times dz$$

In the above expressions by replacing stresses in terms of strains and then in terms of displacements, stress resultants are obtained in terms of displacement. For example,

$$N_x = \int_{-h/2}^{h/2} \sigma_x \left(1 - \frac{Z}{R}\right) dz$$

$$= \frac{E}{1-\mu^2} \int_{-h/2}^{h/2} \left(\varepsilon_x + \mu\varepsilon_\phi\right)\left(1 - \frac{Z}{R}\right) dz$$

$$= \frac{E}{1-\mu^2} \int_{-h/2}^{h/2} \left(\frac{\partial u}{\partial x} - Z\frac{\partial^2 w}{\partial x^2} + \frac{\mu}{R}\frac{\partial v}{\partial \phi} - \frac{\mu}{R}\frac{Z}{R-Z}\frac{\partial^2 w}{\partial \phi^2} - \mu\frac{w}{R-Z}\right)\left(1 - \frac{Z}{R}\right) dz$$

$$= \frac{E}{1-\mu^2} \int_{-h/2}^{h/2} \left[\frac{\partial u}{\partial x} - Z\frac{\partial^2 w}{\partial x^2} + \frac{\mu}{R}\frac{\partial v}{\partial \phi} - \frac{\mu Z}{R^2\left(1 - Z/R\right)}\frac{\partial^2 w}{\partial \phi^2} - \frac{\mu w}{R\left(1 - \frac{Z}{R}\right)}\right]\left(1 - \frac{Z}{R}\right) dz$$

$$= \frac{E}{1-\mu^2} \int_{-h/2}^{h/2} \left[\left(\frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2} + \frac{\mu}{R}\frac{\partial v}{\partial \phi}\left(1 - \frac{Z}{R}\right) - \frac{\mu Z}{R^2}\frac{\partial^2 w}{\partial \phi^2} - \frac{\mu w}{R}\right)\right] dz$$

Now rearranging by collecting terms without z, with z and with z^2, we can write

$$N_x = \frac{E}{1-\mu^2} \int_{-h/2}^{h/2} \left[\left(\frac{\partial u}{\partial x} + \frac{\mu}{R}\frac{\partial v}{\partial \phi} - \frac{\mu w}{R}\right) - \frac{Z}{R}\left(\frac{\partial u}{\partial x} + \frac{\mu}{R}\frac{\partial v}{\partial \phi} - \frac{\mu}{R}\frac{\partial^2 w}{\partial \phi^2}\right) + \frac{Z^2}{R}\frac{\partial^2 w}{\partial x^2}\right] dz$$

Noting that integration in the limits $-\frac{h}{2}$ to $\frac{h}{2}$

(*a*) of 1 is h

(*b*) of z is 0

(*c*) of z^2 is $\frac{h^3}{12}$, we find

$$N_x = \frac{E}{1-\mu^2}\left[\left(\frac{\partial u}{\partial x} + \frac{\mu}{R}\frac{\partial v}{\partial \phi} - \frac{\mu w}{R}\right)h + \frac{h^3}{12R}\frac{\partial^2 w}{\partial x^2}\right]$$

$$= \frac{Eh}{1-\mu^2}\left(\frac{\partial u}{\partial x} + \frac{\mu}{R}\frac{\partial v}{\partial \phi} - \frac{\mu w}{R}\right) + \frac{D}{R}\frac{\partial^2 w}{\partial x^2} \qquad \text{...eqn. 16.19}$$

where $D = \frac{Eh^3}{12(1-\mu^2)}$ is flexural rigidity

To find N_ϕ,

$$N_\phi = \int_{-h/2}^{h/2} \sigma_\phi \, dz$$

$$= \int_{-h/2}^{h/2} \frac{E}{1-\mu^2}(\varepsilon_\phi + \mu\varepsilon_x)dz$$

$$= \frac{E}{1-\mu^2}\int_{-h/2}^{h/2}\left[\left(\frac{1}{R}\frac{\partial v}{\partial \phi} - \frac{Z}{R(R-Z)}\frac{\partial^2 w}{\partial \phi^2} - \frac{w}{R-Z}\right) + \mu\left(\frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2}\right)\right]dz$$

$$= \frac{E}{1-\mu^2}\int_{-h/2}^{h/2}\left[\left(\frac{1}{R}\frac{\partial v}{\partial \phi} + \mu\frac{\partial u}{\partial x}\right) + \frac{1}{R}\left(1 - \frac{R}{R-Z}\right)\frac{\partial^2 w}{\partial \phi^2} - \frac{w}{R-Z} - \mu Z\frac{\partial^2 w}{\partial x^2}\right]dz$$

Note that

$$I_1 = \int_{-h/2}^{h/2}\frac{1}{R-Z} = \left[\log(R-Z)\right]_{-h/2}^{h/2} = \log\left(R + \frac{h}{2}\right) - \log\left(R - \frac{h}{2}\right)$$

$$= \log\left(\frac{R + \frac{h}{2}}{R - \frac{h}{2}}\right) = \log\left(\frac{1 + \frac{h}{2R}}{1 - \frac{h}{2R}}\right)$$

$$= \log\left(1 + \frac{h}{2R}\right) - \log\left(1 - \frac{h}{2R}\right)$$

Noting that $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...$

and $$\log(1-x) = -\left[x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots\right]$$

We find $$I_1 = \frac{h}{2R} - \frac{h^2}{8R^2} + \frac{h^3}{24R^3} - \frac{h^4}{4 \times 64R^4} + \left(\frac{h}{2R} + \frac{h^2}{8R^2} + \frac{h^3}{24R^3} + \frac{h^4}{4 \times 64R^4}\right)$$

$$= \frac{h}{R} + \frac{h^3}{12R^3}.$$

Integration of 1 results into h and integration of Z results into zero.

$\therefore$ $$N_\phi = \frac{E}{1-\mu^2}\left[\left(\frac{1}{R}\frac{\partial v}{\partial \phi} + \mu\frac{\partial u}{\partial x}\right)h + \frac{1}{R}\frac{\partial^2 w}{\partial \phi^2}\left(h - h - \frac{h^3}{12R^2}\right) - w\left(\frac{h}{R} + \frac{h^3}{12R^3}\right)\right]$$

$$= \frac{E}{1-\mu^2}\left[\left(\frac{1}{R}\frac{\partial v}{\partial \phi} + \mu\frac{\partial u}{\partial x} - \frac{w}{R}\right)h - \left(\frac{\partial^2 w}{\partial \phi^2} + w\right)\frac{h^3}{12R^3}\right]$$

$$= \frac{Eh}{1-\mu^2}\left[\frac{1}{R}\frac{\partial v}{\partial \phi} + \mu\frac{\partial u}{\partial x} - \frac{w}{R}\right] - \frac{D}{R^3}\left(w + \frac{\partial^2 w}{\partial \phi^2}\right) \quad \text{...eqn. 16.20}$$

Proceeding on the same line all stress resultants can be obtained as shown below:

$$N_x = \frac{Eh}{1-\mu^2}\left[\frac{\partial u}{\partial x} + \frac{\mu}{R}\frac{\partial v}{\partial \phi} - \mu\frac{w}{R}\right] + \frac{D}{R}\frac{\partial^2 w}{\partial x^2}$$

$$N_\phi = \frac{Eh}{1-\mu^2}\left[\frac{1}{R}\frac{\partial v}{\partial \phi} - \frac{w}{R} + \mu\frac{\partial u}{\partial x}\right] - \frac{D}{R^3}\left(w + \frac{\partial^2 w}{\partial \phi^2}\right)$$

$$N_{\phi x} = \frac{Eh}{1-\mu^2}\left(\frac{1-\mu}{2}\right)\left[\frac{1}{R}\frac{\partial u}{\partial \phi} + \frac{\partial v}{\partial x}\right] - \frac{D}{R^2}\frac{1-\mu}{2}\left(\frac{\partial u}{R\partial \phi} + \frac{\partial^2 w}{\partial x \partial \phi}\right)$$

$$N_{x\phi} = \frac{Eh}{1-\mu^2}\frac{1-\mu}{2}\left[\frac{1}{R}\frac{\partial u}{\partial \phi} + \frac{\partial v}{\partial x}\right] - \frac{D}{R^2}\frac{1-\mu}{2}\left(\frac{\partial v}{\partial x} - \frac{1}{R}\frac{\partial^2 w}{\partial x \partial \phi}\right)$$

$$M_x = -D\left[\frac{\partial^2 w}{\partial x^2} + \frac{\mu}{R^2}\frac{\partial^2 w}{\partial \phi^2} + \frac{1}{R}\frac{\partial v}{\partial x} + \frac{\mu}{R^2}\frac{\partial v}{\partial \phi}\right]$$

$$M_\phi = -\frac{D}{R^2}\left(w + \frac{\partial^2 w}{\partial \phi^2} + \mu R^2\frac{\partial^2 w}{\partial x^2}\right)$$

$$M_{\phi x} = -D(1-\mu)\left[\frac{1}{R}\frac{\partial^2 w}{\partial x \partial \phi} - \frac{1}{2R^2}\frac{\partial u}{\partial \phi} + \frac{1}{2R}\frac{\partial v}{\partial x}\right]$$

$$M_{x\phi} = -D(1-\mu)\left[\frac{1}{R}\frac{\partial^2 w}{\partial x \partial \phi} + \frac{1}{R}\frac{\partial v}{\partial x}\right]$$

The above relations are known as **'Stress Resultants to Flugge's Accuracy'**.

16.6 SIMPLIFIED STRESS RESULTANTS AND DISPLACEMENT RELATIONS

As it is very difficult to make use of the stress resultants to Flugge's accuracy, researchers made assumptions to form shell equations. Usual assumptions are (*i*) to consider the strains across a section uniform (*ii*) take the value of Poisson's ratio $\mu = 0$. Finsterwalder and Schorer went a step forward and assumed $M_x = 0$ and $M_{x\phi} = 0$. Table 16.1 shows the simplified stress resultant – displacement relations assumed by different researchers. It may be noted that except Holland, other three have omitted the terms which do not figure in 'Disk action, Plate action and Membrane action'.

16.7 DKJ THEORY

The assumptions $M_x = M_{x\phi} = 0$, make Finsterwalder and Schorer theories applicable only to long shells. Holland and DKJ theories can be applied to all classes of shells. However, DKJ theory is commonly used shell theory.

In 1933-34, Donnel developed the theory for his studies on stability of thin walled circular cylinders. Kármán and Tsien employed the same theory in 1941 for their investigations on the buckling of cylindrical shells. In 1947, Jenkins published a book in which he presented the theory developed by Donnel and Kármán in the form suitable for the analysis of cylindrical shell roof. Hence, the theory in now known as Donnel-Kármán-Jenkins in short DKJ theory.

Table 16.1 Simplified stress resultants—Displacement relations

Stress resultant	*Finsterwalder*	*Schorer*	*Holland*	*DKJ*
N_x	$Eh\frac{\partial u}{\partial x}$	$Eh\frac{\partial v}{\partial x}$	$Eh\frac{\partial v}{\partial x}$	$Eh\frac{\partial v}{\partial x}$
N_ϕ	$Eh\left(\frac{\partial v}{R\partial\phi} - \frac{w}{R}\right)$	$Eh\left(\frac{\partial v}{R\partial\phi} - \frac{w}{R}\right)$	$Eh\left(\frac{\partial v}{R\partial\phi} - \frac{w}{R}\right)$	$Eh\left(\frac{\partial v}{R\partial\phi} - \frac{w}{R}\right)$
$N_{x\phi} = N_{\phi x}$	$\frac{Eh}{2}\left(\frac{1}{R}\frac{\partial u}{\partial\phi} + \frac{\partial v}{\partial x}\right)$	$\frac{Eh}{2}\left(\frac{1}{R}\frac{\partial u}{\partial\phi} + \frac{\partial v}{\partial x}\right)$	$\frac{Eh}{2}\left(\frac{1}{R}\frac{\partial u}{\partial\phi} + \frac{\partial v}{\partial x}\right)$	$\frac{Eh}{2}\left(\frac{\partial u}{R\partial\phi} + \frac{\partial v}{\partial x}\right)$
M_x	0	0	$-D\frac{\partial^2 w}{\partial x^2}$	$-D\frac{\partial^2 w}{\partial x^2}$
M_ϕ	$-\frac{D}{R^2}\frac{\partial^2 w}{\partial\phi^2}$	$-\frac{D}{R^2}\frac{\partial^2 w}{\partial\phi^2}$	$-D\left(\frac{1}{R^2}\frac{\partial^2 w}{\partial\phi^2} + w\right)$	$-\frac{D}{R^2}\frac{\partial^2 w}{\partial\phi^2}$
$M_{x\phi} = M_{\phi x}$	0	0	$-D\left(\frac{1}{R}\frac{\partial^2 w}{\partial x\partial\phi} + \frac{1}{R}\frac{\partial v}{\partial x}\right)$	$-\frac{D}{R}\frac{\partial^2 w}{\partial x\partial\phi}$

In this theory (and also in Shorer's theory) the term Q_ϕ appearing in equation of equilibrium 16.2 is dropped, since such term do not appear in the corresponding equation of equilibrium of disc, plate or the membrane shell. Hence, the equation is rewritten as,

$$\frac{\partial N_{x\phi}}{\partial x}+\frac{1}{R}\frac{\partial N_{\phi}}{\partial \phi}+Y=0.$$

Since, in this theory, $N_{x\phi}$ is taken as equal to $N_{\phi x}$, equation of equilibrium 16.6 shows $M_{x\phi} = 0$. But in this theory $M_{x\phi}$ is taken as $-\frac{D}{R}\frac{\partial^2 w}{\partial x \partial \phi}$. Thus, these is clear violation of fundamental principle of mechanics. However, still this theory is used, since, $M_{x\phi}$ is generally small quantity.

Since, $N_{x\phi} = N_{\phi x}$ and $M_{x\phi} = M_{\phi x}$, the theory is developed below using $N_{x\phi}$ for shearing forces and $M_{x\phi}$ for twisting moment.

We have, now, the following equilibrium equations.

$$\frac{\partial N_x}{\partial x}+\frac{1}{R}\frac{\partial N_{x\phi}}{\partial \phi}+X=0 \qquad ...(1)$$

$$\frac{\partial N_{x\phi}}{\partial x}+\frac{1}{R}\frac{\partial N_{\phi}}{\partial \phi}+Y=0 \qquad ...(2)$$

$$\frac{\partial Q_x}{\partial x}+\frac{1}{R}\frac{\partial Q_{\phi}}{\partial \phi}+\frac{N_{\phi}}{R}+Z=0 \qquad ...(3)$$

$$\frac{\partial M_x}{\partial x}+\frac{1}{R}\frac{\partial M_{x\phi}}{\partial \phi}-Q_x=0 \qquad ...(4)$$

$$\frac{\partial M_{x\phi}}{\partial x}+\frac{1}{R}\frac{\partial M_{\phi}}{\partial \phi}-Q_{\phi}=0 \qquad ...(5)$$

The following are the stress-resultant and displacement relations:

$$N_x = Eh\frac{\partial u}{\partial x} \qquad ...(6)$$

$$N_{\phi} = Eh\left(\frac{\partial u}{R\partial \phi}-\frac{w}{R}\right) \qquad ...(7)$$

$$N_{x\phi} = N_{\phi x} = \frac{Eh}{2}\left(\frac{1}{R}\frac{\partial u}{\partial \phi}+\frac{\partial v}{\partial x}\right) \qquad ...(8)$$

$$M_x = -D\frac{\partial^2 w}{\partial x^2} \qquad ...(9)$$

$$M_{\phi} = -D\frac{1}{R^2}\frac{\partial^2 w}{\partial \phi^2} \qquad ...(10)$$

and

$$M_{x\phi} = M_{\phi x} = -\frac{D}{R}\frac{\partial^2 w}{\partial x \partial \phi} \qquad ...(11)$$

Thus, we have 11 equations and number of unknowns are—

$$N_x,\ N_\phi,\ N_{x\phi} = 3$$
$$Q_x,\ Q_\phi = 2$$
$$M_x,\ M_\phi,\ M_{x\phi} = 3$$
$$u,\ v \text{ and } w = 3$$
$$\text{Total} \quad 11$$

Hence, the problem can be solved.

From equation (5),

$$Q_\phi = \frac{\partial M_{x\phi}}{\partial x} + \frac{1}{R}\frac{\partial M_\phi}{\partial \phi}$$

$$= -\frac{D}{R}\frac{\partial^3 w}{\partial x^2 \partial \phi} - \frac{D}{R^3}\frac{\partial^3 w}{\partial \phi^3}$$

$$= -\frac{D}{R}\frac{\partial}{\partial \phi}\left(\frac{\partial^2 w}{\partial x^2} + \frac{1}{R^2}\frac{\partial^2 w}{\partial \phi^2}\right)$$

$$= -\frac{D}{R}\frac{\partial}{\partial \phi}\left(\nabla^2 w\right) \qquad \text{...(12)}$$

From eqn. (4),

$$Q_x = \frac{\partial M_x}{\partial x} + \frac{1}{R}\frac{\partial M_{x\phi}}{\partial \phi}$$

$$= -D\frac{\partial^3 w}{\partial x^3} + \frac{1}{R}\left(-\frac{D}{R}\right)\frac{\partial^3 w}{\partial x \partial \phi^2}$$

$$= -D\left[\frac{\partial^3 x}{\partial x^3} + \frac{1}{R^2}\frac{\partial^3 w}{\partial x \partial \phi^2}\right]$$

$$= -D\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} + \frac{1}{R^2}\frac{\partial^2 w}{\partial \phi^2}\right)$$

$$= -D\frac{\partial}{\partial x}\left(\nabla^2 w\right) \qquad \text{...(13)}$$

From eqn. (3),

$$N_\phi = -R\frac{\partial Q_x}{\partial x} - \frac{\partial Q_\phi}{\partial \phi} - ZR$$

$$= +R \cdot D\frac{\partial^2\left(\nabla^2 w\right)}{\partial x^2} + \frac{D}{R}\frac{\partial^2\left(\nabla^2 w\right)}{\partial \phi^2} - ZR$$

$$= DR\left[\frac{\partial^2(\nabla^2 w)}{\partial x^2} + \frac{1}{R^2}\frac{\partial^2}{\partial \phi^2}(\nabla^2 w)\right] - ZR$$

$$= DR\left(\frac{\partial^2}{\partial x^2} + \frac{1}{R^2}\frac{\partial^2}{\partial \phi^2}\right)(\nabla^2 w) - ZR$$

$$= DR\nabla^2(\nabla^2 w) - ZR$$

$$= DR(\nabla^4 w) - ZR \qquad ...(14)$$

From eqn. (2),

$$\frac{\partial N_{x\phi}}{\partial x} = -\frac{\partial N_\phi}{R\partial \phi} - Y$$

$$= -D\frac{\partial}{\partial \phi}(\nabla^4 w) + \frac{\partial Z}{\partial \phi} - Y \qquad ...(15)$$

From eqn. (1),

$$\frac{\partial N_x}{\partial x} = -\frac{\partial N_{x\phi}}{R\partial \phi} - X$$

$$\therefore \qquad \frac{\partial^2 N_x}{\partial x^2} = -\frac{\partial^2 N_{x\phi}}{R\partial x\partial \phi} - \frac{\partial X}{\partial x}$$

$$= \frac{D}{R}\frac{\partial^2(\nabla^4 w)}{\partial \phi^2} - \frac{\partial^2 Z}{R\partial \phi^2} + \frac{1}{R}\frac{\partial Y}{\partial \phi} - \frac{\partial X}{\partial x} \qquad ...(16)$$

Thus, now we have expressed all stress resultants in only one unknown '*w*'. To form compatibility equations consider the force displacement equations 6, 7 and 8.

$$N_x = Eh\frac{\partial u}{\partial x} \qquad ...(6)$$

$$N_\phi = Eh\left(\frac{1}{R}\frac{\partial u}{\partial \phi} - \frac{w}{R}\right) \qquad ...(7)$$

and

$$N_{x\phi} = \frac{Eh}{2}\left(\frac{1}{R}\frac{\partial u}{\partial \phi} + \frac{\partial v}{\partial x}\right) \qquad ...(8)$$

To eliminate v in the above equations:

$$\frac{\partial N_\phi}{\partial x} = Eh\left(\frac{1}{R}\frac{\partial^2 v}{\partial x\partial \phi} - \frac{1}{R}\frac{\partial w}{\partial x}\right) \qquad ...(a)$$

$$2\frac{1}{R}\frac{\partial N_{x\phi}}{\partial \phi} = Eh\left(\frac{1}{R^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{1}{R}\frac{\partial^2 v}{\partial x\partial \phi}\right) \qquad ...(b)$$

Subtracting eqn. (*b*) from eqn. (*a*), we get,

$$\frac{\partial N_\phi}{\partial x} - 2\frac{1}{R}\frac{\partial N_{x\phi}}{\partial \phi} = -Eh\frac{1}{R}\frac{\partial w}{\partial x} - Eh\frac{1}{R^2}\frac{\partial^2 u}{\partial \phi^2}$$

$$= -Eh\left[\frac{1}{R}\frac{\partial w}{\partial x} + \frac{1}{R^2}\frac{\partial^2 u}{\partial \phi^2}\right] \quad ...(17)$$

To eliminate *u* from eqn. (6) and (17):

From eqn. 6, $$\frac{1}{R^2}\frac{\partial^2 N_x}{\partial \phi^2} = \frac{1}{R^2}\cdot Eh\frac{\partial^3 u}{\partial x \partial \phi^2} \quad ...(c)$$

From eqn. 17, $$\frac{\partial^2 N_\phi}{\partial x^2} - \frac{2}{R}\cdot\frac{\partial^2 N_{x\phi}}{\partial x \partial \phi} = -Eh\left[\frac{\partial^2 w}{R\partial x^2} + \frac{1}{R^2}\frac{\partial^3 u}{\partial x \partial \phi^2}\right] \quad ...(d)$$

Adding eqn. (*c*) and eqn. (*d*) we get,

$$\frac{\partial^2 N_\phi}{\partial x^2} + \frac{1}{R^2}\frac{\partial^2 N_x}{\partial \phi^2} - \frac{2}{R}\frac{\partial^2 N_{x\phi}}{\partial x \partial \phi} = -Eh\frac{\partial^2 w}{R\partial x^2}$$

Differentiating both sides twice w.r.t. *x* we get,

$$\frac{\partial^4 N_\phi}{\partial x^4} + \frac{1}{R^2}\frac{\partial^4 N_x}{\partial x^2 \partial \phi^2} - \frac{2}{R}\frac{\partial^4 N_{x\phi}}{\partial x^3 \partial \phi} = -\frac{Eh}{R}\frac{\partial^4 w}{\partial x^4}$$

Substituting the values of N_ϕ, $\frac{\partial N_{x\phi}}{\partial x}$ and $\frac{\partial^2 N_x}{\partial x^2}$ from eqn. 14, 15 and 16, we get equations.

$$DR\frac{\partial^4(\nabla^4 w)}{\partial x^4} - R\frac{\partial^4 Z}{\partial x^4} + \frac{1}{R^2}\left[\frac{D}{R}\frac{\partial^4(\nabla^4 w)}{\partial \phi^4} - \frac{\partial^4 Z}{R\partial \phi^4} + \frac{1}{R}\frac{\partial^3 Y}{\partial \phi^3} - \frac{\partial^3 X}{\partial x \partial \phi^2}\right]$$

$$-\frac{2}{R}\left[-D\frac{\partial^4(\nabla^4 w)}{\partial x^2 \partial y^2} + \frac{\partial^4 Z}{\partial x^2 \partial \phi^2} - \frac{\partial^3 Y}{\partial x^2 \partial \phi}\right] = -\frac{Eh}{R}\frac{\partial^4 w}{\partial x^4}$$

Dividing throughout by *R* and bringing load terms to right hand side, we get

$$D\left[\frac{\partial^4(\nabla^4 w)}{\partial x^4} + \frac{2}{R^2}\frac{\partial^4(\nabla^4 w)}{\partial x^2 \partial \phi^2} + \frac{\partial^4(\nabla^4 w)}{R^4 \partial \phi^4}\right] + \frac{Eh}{R^2}\frac{\partial^4 w}{\partial x^4}$$

$$= \left[\frac{\partial^4 Z}{\partial x^4} + \frac{1}{R^4}\frac{\partial^4 Z}{\partial \phi^4} + \frac{2}{R^2}\frac{\partial^4 Z}{\partial x^2 \partial \phi^2}\right] - \frac{1}{R^4}\frac{\partial^3 Y}{\partial \phi^3} - \frac{2}{R^2}\frac{\partial^3 Y}{\partial x^2 \partial \phi} + \frac{1}{R^2}\frac{\partial^3 X}{\partial x \partial \phi^2}$$

i.e. $$D\nabla^4(\nabla^4 w) + Eh\frac{\partial^4 w}{\partial x^4} = \nabla_Z^4 - \frac{1}{R^4}\frac{\partial^3 Y}{\partial \phi^3} - \frac{2}{R^2}\frac{\partial^3 Y}{\partial x^2 \partial \phi} + \frac{1}{R^2}\frac{\partial^3 X}{\partial x \partial \phi^2}$$

i.e.
$$\nabla^8 w + \frac{Eh}{R^2 D}\frac{\partial^4 w}{\partial x^2} = \frac{1}{D}\left[\nabla_Z^4 = \frac{1}{R^4}\frac{\partial^3 Y}{\partial \phi^3} - \frac{2}{R^2}\frac{\partial^3 Y}{\partial x^2 \partial \phi} + \frac{1}{R^2}\frac{\partial^3 X}{\partial x \partial \phi^2}\right]$$

where $\nabla_w^8 = \nabla^4(\nabla_w^4)$

$$= \left(\frac{\partial^4}{\partial x^4} + \frac{2\partial^4}{R^2 \partial x^2 \partial \phi^2} + \frac{\partial^4}{R^4 \partial^4 \phi}\right)\left(\frac{\partial^4 w}{\partial x^4} + \frac{2}{R^2}\frac{\partial^4 w}{\partial x^2 \partial \phi^2} + \frac{\partial^4 w}{R^4 \partial \phi^4}\right)$$

$$= \frac{\partial^8 w}{\partial x^8} + \frac{4\partial^8 w}{R^2 \partial x^6 \partial \phi^2} + \frac{6\partial^8 w}{R^4 \partial x^4 \partial \phi^4} + \frac{4\partial^8 w}{R^6 \partial x^2 \partial \phi^6} + \frac{\partial^8 w}{R^8 \partial \phi^8}$$

Since $\mu = 0,\ D = \frac{Eh}{1-\mu^2} = Eh.$

$\therefore$ The eqn. of shell is,

$$\nabla^8 \mathbf{w} + \frac{\mathbf{h}}{\mathbf{R^2 I}}\frac{\partial^4 \mathbf{w}}{\partial \mathbf{x}^4} = \frac{\mathbf{1}}{\mathbf{D}}\left[\nabla_Z^4 - \frac{\mathbf{1}}{\mathbf{R}}\left(\frac{\mathbf{1}}{\mathbf{R^3}}\frac{\partial^3 \mathbf{Y}}{\partial \phi^3} + \frac{\mathbf{2}}{\mathbf{R}}\frac{\partial^3 \mathbf{Y}}{\partial \mathbf{x}^2 \partial \phi}\right) + \frac{\mathbf{1}}{\mathbf{R^2}}\frac{\partial^3 \mathbf{X}}{\partial \mathbf{x} \partial \phi^2}\right] \quad \text{...eqn. 16.21}$$

Equation 16.21 may be called as shell equation as per DKJ theory.

The solution of the above equation for given loading gives expression for *w* at any point and knowing '*w*', any stress resultant may be found.

16.8 DETERMINATION OF PARTICULAR INTEGRAL FOR DEAD LOAD

Consider the case of dead load *g*/unit surface area. Expanding it in the form of Fourier series we get,

$$g = g \sum_{m=1,3,...}^{\infty} \frac{4}{m\pi}\cos\frac{n\pi x}{L}$$

Taking only first term of loading, we get,

$$g \approx \frac{4g}{m\pi}\cos\frac{\pi x}{L}$$

$$= g'\cos kx \text{ where } g' = \frac{4g}{m\pi} \text{ and } k = \frac{\pi}{L}$$

Dead load components are,

$$X = 0,\ Y = g'\sin\phi\cos kx \text{ and } z = g'\cos\phi\cos kx.$$

Let *w* be in the form,

$$w = Cg'\cos\phi\cos kx$$

Then from plate equation, we get,

$$\nabla^8 w + \frac{h}{R^2 I}\frac{\partial^4 w}{\partial x^4} = \frac{1}{D}\left[\left(k^4 + \frac{2k^2}{R^2} + \frac{1}{R^4}\right) - \frac{1}{R}\left(-\frac{1}{R^3} - \frac{2}{R}k^2\right)\right]g'\cos\phi\cos kx$$

i.e. $$\left(\frac{\partial^2}{\partial x^2}+\frac{1}{R^2}\frac{\partial^2}{\partial \phi^2}\right)^4 w+\frac{h}{R^2 I}\frac{\partial^4 w}{\partial x^4}=\frac{1}{D}\left[k^4+\frac{2k^2}{R^2}+\frac{1}{R^4}+\frac{1}{R^4}+\frac{2k^2}{R^2}\right]g'\cos kx\cos\phi$$

i.e. $$\left[\left(k^2+\frac{1}{R^2}\right)^4+\frac{h}{R^2 I}k^4\right]C=\frac{1}{D}\left[k^4+\frac{4k^2}{R^2}+\frac{2}{R^4}\right]$$

$\therefore$ Multiplying both sides by R^8, we get

$$\left[\left(k^2R^2+1\right)^4+\frac{hR^6}{I}k^4\right]C=\frac{R^4}{D}\left[k^4R^4+4k^2R^2+2\right]$$

$\therefore$ $$C=\frac{R^4}{D}\frac{k^4R^4+4k^2R^2+2}{\left(1+k^2R^2\right)^4+k^4R^4\frac{h}{I}R^2}$$ ...eqn. 16.22

In an analysis problem, all terms in RHS of eqn. 16.22 are known. Hence, C is found and hence, w is found. Substituting the expression of w in eqns. 6 to 16, expressions for all stress resultants and displacements can be determined. These expressions are given below:

$$M_\phi=\frac{EI}{R^2}Cg'\cos\phi\cos kx$$

$$M_x=EIk^2Cg'\cos\phi\cos kx$$

$$M_{x\phi}=M_{\phi x}=-\frac{EIk}{R}Cg'\sin\phi\sin kx$$

$$Q_x=-EI\left(k^3+\frac{k^2}{R}\right)Cg'\cos\phi\cdot\sin kx$$

$$Q_\phi=-EI\left(\frac{1}{R^3}+\frac{k^2}{R}\right)Cg'\sin\phi\cos kx$$

$$N_\phi=R\left(H_1-1\right)g'\cos\phi\cos kx$$

where $$H_1=EI\left(k^4+\frac{2k^2}{R^2}+\frac{1}{R^4}\right)C+2$$

$$N_{x\phi}=N_{\phi x}=\frac{H_1}{k}g'\sin\phi\sin kx$$

$$N_x=\frac{H_1}{k^2R}g'\cos\phi\cos kx$$

$$u=\frac{H_1}{Ehk^3R}g'\cos\phi\cos kx$$

$$v = -\frac{H_1 H_2}{k} g' \sin\phi \sin kx$$

where $$H_2 = \frac{1}{Ehk}\left[2 + \frac{1}{k^2 R^2}\right]$$

and $$w = Cg' \cos\phi \cos kx \qquad \text{...eqn. 16.23}$$

If for any particular shell if the above values are calculated, it will be found that M_x, M_ϕ, $M_{x\phi}$, Q_x and Q_ϕ values are very small (hardly 0.3 percent of final values) and N_x, N_ϕ, $N_{x\phi}$ values are almost equal to those obtained from membrane theory. Hence, **we replace particular solution by membrane solution.**

16.9 HOMOGENEOUS SOLUTION

For this part, the equation is,

$$\nabla^8 w + \frac{h}{R^2 I}\frac{\partial^4 w}{\partial x^4} = 0$$

The solution consists of two parts as shown below:

$$w = f(\phi)\cos\frac{n\pi x}{L} + f(x)\cos\frac{n\phi}{\phi_k}.$$

The first part gives the disturbances from the edge and the second part gives the disturbances from the traverses. As the equation is of the eighth order, there will be eight constants in the first term and eight in the second term. If the boundary conditions at traverses are assumed as simply supported only first term may be considered. In that case

$$w = f(\phi)\cos\frac{n\pi x}{L}$$

$$= He^{m\phi}\cos\frac{n\pi x}{L}$$

considering only first term of '*w*',

$$w = He^{m\phi}\cos kx \text{ where } k = \frac{\pi}{L}$$

The equation reduces to the form,

$$\left(-k^2 + \frac{1}{R^2}m^2\right)^4 + \frac{h}{R^2 I}k^4 = 0$$

or $$\left(m^2 - k^2 R^2\right)^4 + \frac{hR^6}{I}k^4 = 0.$$

Substituting $$\rho^8 = \frac{hR^6}{I} k^4$$

$$= \frac{hR^6}{\frac{1}{12} \times 1 \times h^3} k^4 = \frac{12 R^6 k^4}{h^2},$$

We get, $$\left(m^2 - k^2 R^2\right)^4 + \rho^8 = 0$$

or $$\left(m^2 - k^2 R^2\right)^4 = -\rho^8$$

i.e. $$\left(\frac{m^2 - k^2 R^2}{\rho^2}\right)^4 = -1 = \cos(2n\pi + \pi) + i \sin(2n\pi + \pi)$$

$\therefore$ $$\frac{m^2 - k^2 R^2}{\rho^2} = \cos\frac{2n\pi + \pi}{4} + i \sin\frac{(2n\pi + \pi)}{4} \quad \text{where } n = 0, 1, 2, 3, \ldots$$

$$= \pm \frac{1}{\sqrt{2}} (1 \pm i)$$

$\therefore$ $$m^2 = \rho^2 \left[\frac{k^2 R^2}{\rho^2} \pm \frac{1}{\sqrt{2}} (1 \pm i)\right]$$

$$= \frac{\rho^2}{\sqrt{2}} \left[\frac{k^2 R^2}{\rho^2} \sqrt{2} \pm 1 \pm i\right]$$

$$= \frac{\rho^2}{\sqrt{2}} [\gamma \pm 1 \pm i]$$

where $$\gamma = \frac{k^2 R^2}{\rho^2} \sqrt{2}$$

$\therefore$ $$m = \pm \frac{\rho}{\sqrt[4]{2}} [\gamma \pm 1 \pm i]^{1/2}$$

The eight roots m_1, m_2, ..., m_8 may now be written as,

$$\begin{aligned} m_1 &= \alpha_1 + i\beta_1 & m_5 &= -m_1 \\ m_2 &= \alpha_1 - i\beta_1 & m_6 &= -m_2 \\ m_3 &= \alpha_2 + i\beta_2 & m_7 &= -m_3 \\ m_4 &= \alpha_2 - i\beta_2 & m_8 &= -m_4 \end{aligned} \qquad \ldots(a)$$

where

$$\alpha_1 = \frac{\rho}{\sqrt[4]{2}}\left[\frac{\sqrt{(1+\gamma)^2+1}+1+\gamma}{2}\right]^{1/2}$$

$$\alpha_2 = \frac{\rho}{\sqrt[4]{2}}\left[\frac{\sqrt{(1-\gamma)^2+1}-(1-\gamma)}{2}\right]^{1/2}$$

$$\beta_1 = \frac{\rho}{\sqrt[4]{2}}\left[\frac{\sqrt{(1+\gamma)^2+1}-(1+\gamma)}{2}\right]^{1/2}$$

and

$$\beta_2 = \frac{\rho}{\sqrt[4]{2}}\left[\frac{\sqrt{(1-\gamma)^2+1}+(1-\gamma)}{2}\right]^{1/2} \qquad ...(b)$$

Thus,

$$w = \left[H_1 e^{m_1\phi} + H_2 e^{m_2\phi} + ... + H_8 e^{m_8\phi}\right]\cos kx \qquad ...(c)$$

where $H_1, H_2, ..., H_8$ are arbitrary constants and are to be determined from boundary conditions at edges.

Substituting for $m_1, m_2, ..., m_8$ in terms of α and β (from eqn. (*a*)), we get,

$$w = \Big[H_1 e^{(\alpha_1+i\beta_1)\phi} + H_2 e^{(\alpha_1-i\beta_1)\phi} + H_3 e^{(\alpha_2+i\beta_2)\phi} + H_4 e^{(\alpha_2-i\beta_2)\phi}$$

$$+H_5 e^{-(\alpha_3+i\beta_3)\phi} + H_6 e^{-(\alpha_1-i\beta_1)\phi} + H_7 e^{-(\alpha_2+i\beta_2)\phi} + H_8 e^{-(\alpha_2-i\beta_2)\phi}\Big]\cos kx$$

$$= \Big[H_1 e^{\alpha_1\phi}(\cos\beta_1\phi + i\sin\beta_1\phi) + H_2 e^{\alpha_1\phi}(\cos\beta_1\phi - i\sin\beta_1\phi)$$

$$+H_3 e^{\alpha_2\phi}(\cos\beta_2\phi + i\sin\beta_2\phi) + H_4 e^{\alpha_2\phi}(\cos\beta_2\phi - i\sin\beta_2\phi)$$

$$+H_5 e^{-\alpha_1\phi}(\cos\beta_1\phi + i\sin\beta_1\phi) + H_6 e^{-\alpha_2\phi}(\cos\beta_1\phi - i\sin\beta_1\phi)$$

$$+H_7 e^{-\alpha_2\phi}(\cos\beta_2\phi + i\sin\beta_2\phi) + H_8 e^{-\alpha_2\phi}(\cos\beta_2\phi - i\sin\beta_2\phi)\Big]\cos kx$$

$$= \Big[e^{\alpha_1\phi}(H_1+H_2)\cos\beta_1\phi + i(H_1-H_2)e^{\alpha_1\phi}\sin\beta_1\phi$$

$$+ e^{\alpha_2\phi}(H_3+H_4)\cos\beta_2\phi + i(H_3-H_4)e^{\alpha_2\phi}\sin\beta_2\phi$$

$$+ e^{-\alpha_1\phi}(H_5+H_6)\cos\beta_1\phi + i(H_5-H_6)e^{-\alpha_1\phi}\sin\beta_1\phi$$

$$+ e^{-\alpha_2\phi}(H_7+H_8)\cos\beta_2\phi + i(H_7-H_8)e^{-\alpha_2\phi}\sin\beta_2\phi\Big]\cos kx$$

It is to be noted that the arbitrary constants $H_1, H_2, ..., H_8$ are complex numbers. Since, w is real, it follows that $(H_1 + H_2)$, $i\,(H_1 - H_2)$, ..., $i\,(H_7 - H_8)$ should be real. It means that H_1 and H_2, H_3 and H_4, H_5 and H_6, H_7 and H_8 should be *conjugate pairs.* Introducing real constants as,

$A_1 = H_1 + H_2$, $A_2 = i\,(H_1 - H_2)$, $A_3 = H_3 + H_4$, $A_4 = i\,(H_3 - H_4)$
$A_5 = H_5 + H_6$, $A_6 = i\,(H_5 - H_6)$, $A_7 = H_7 + H_1$ and $A_8 = i\,(H_7 - H_6)$,

we get,

$$w = \left[A_1 e^{\alpha_1\phi}\cos\beta_1\phi + A_2 e^{\alpha_1\phi}\sin\beta_1\phi + A_3 e^{\alpha_2\phi}\cos\beta_2\phi + A_4 e^{\alpha_2\phi}\sin\beta_2\phi\right.$$
$$\left. + A_5 e^{-\alpha_1\phi}\cos\beta_1\phi + A_6 e^{-\alpha_1\phi}\sin\beta_1\phi + A_7 e^{-\alpha_2\phi}\cos\beta_2\phi + A_8 e^{-\alpha_2\phi}\sin\beta_2\phi\right]\cos kx$$

Let $\quad \dfrac{A_1 + A_5}{2} = C_1$ and $\dfrac{A_1 - A_5}{2} = C_2$

then, $\quad A_1 = C_1 + C_2,\ A_5 = C_1 - C_2$

$\therefore \quad A_1 e^{\alpha_1\phi}\cos\beta_1\phi + A_2 e^{-\alpha_1\phi}\cos\beta_1\phi$

$$= \left[(C_1 + C_2)e^{\alpha_1\phi} + (C_1 - C_2)e^{-\alpha_1\phi}\right]\cos\beta_1\phi$$

$$= C_1\left(e^{\alpha_1\phi} + e^{-\alpha_1\phi}\right)\cos\beta_1\phi + C_2\left(e^{\alpha_1\phi} - e^{-\alpha_1\phi}\right)\cos\beta_1\phi$$

$$= 2C_1 \cosh\alpha_1\phi \cos\beta_1\phi + 2C_2 \sinh\alpha_1\phi \cos\beta_1\phi$$

Similarly, substituting $A_3 = C_3 + C_4$, $A_4 = C_3 - C_4$, $A_5 = C_5 + C_6$, $A_6 = C_5 - C_6$, $A_7 = C_7 + C_8$ and $A_8 = C_7 - C_8$,
we get,

$$\begin{aligned} w = 2[&C_1 \cosh\alpha_1\phi \cos\beta_1\phi + C_2 \sinh\beta_1\phi \cos\beta_1\phi \\ &+ C_3 \cosh\alpha_1\phi \sin\beta_1\phi + C_4 \sinh\alpha_1\phi \cdot \sin\beta_1\phi \\ &+ C_5 \cosh\alpha_2\phi \cos\beta_2\phi + C_6 \sinh\beta_2\phi \cos\beta_2\phi \\ &+ C_7 \cosh\alpha_2\phi \sin\beta_2\phi + C_8 \sinh\alpha_2\phi \sin\beta_2\phi] \cos kx \end{aligned} \qquad ...(16.24)$$

For symmetric loading, anti-symmetric terms should vanish. Hence, displacement for symmetric case is,

$$\begin{aligned} w &= 2[C_1 \cosh\alpha_1\phi \cos\beta_1\phi + C_4 \sinh\alpha_1\phi \sin\beta_1\phi \\ &\quad + C_5 \cosh\alpha_2\phi \cos\beta_2\phi + C_8 \sinh\alpha_2\phi \sin\beta_2\phi] \cos kx \\ &= 2\,(a \cos\beta_1\phi \cosh\alpha_1\phi - b \sin\beta_1\phi \sinh\alpha_1\phi \\ &\quad + c \cos\beta_2\phi \cosh\alpha_2\phi - d \sin\beta_2\phi \sinh\beta_2\phi) \cos kx \end{aligned} \qquad ...(16.25)$$

where $\quad C_1 = a,\ C_2 = -b,\ C_3 = c$ and $C_4 = -d.$

Substituting the value of w in the stress resultant expressions 16.23, all stress resultants can be expressed in 'w'. In Table 16.2 and 16.3, they are arranged in the convenient form in which,

F – represent stress resultant/displacement a, b, c and d are arbitrary constants

$$\left.\begin{aligned} Q'_x &= Q_x + \frac{\partial M_x}{\partial x} \\ Q'_\phi &= Q_\phi + \frac{\partial M_\phi}{R\partial\rho} \end{aligned}\right\} \text{Vertical reactions}$$

$$m_1 = \frac{\alpha_1\sqrt[4]{2}}{\rho} \qquad m_2 = \alpha_2 \cdot \frac{\sqrt[4]{2}}{\rho}$$

$$n_1 = \frac{\beta_1\sqrt[4]{2}}{\rho} \qquad n_2 = \frac{\beta_2\sqrt[4]{2}}{\rho}$$

Table 16.2 Stress resultants and displacements due to complementary function (Even)

$F = \bar{R}$ $[(aB_1 - bB_2)\cos\beta_1\phi \cosh \alpha_1\phi - (aB_2 + bB_1)\sin\beta_1\phi \sinh \alpha_1\phi$
$+(cB_3 - dB_4)\cos\beta_2\phi \cosh \alpha_2\phi - (cB_4 + dB_3)\sin\beta_2\phi \sinh \alpha_2\phi]$

F	$\bar{R}$	B_1	B_2	B_3	B_4
M_ϕ	$\frac{2EI}{R^2}\cos kx$	$\frac{\rho^2}{\sqrt{2}}(1+\gamma)$	$\frac{\rho^2}{\sqrt{2}}$	$\frac{\rho^2}{\sqrt{2}}(\gamma-1)$	$\frac{\rho^2}{\sqrt{2}}$
M_x	$-2EIk^2\cos kx$	1	0	1	0
Q_x	$-\frac{2EIk^3}{\gamma}\sin kx$	1	1	-1	1
N_ϕ	$-2EI\frac{2k^4}{\gamma^2}\cos kx$	0	1	0	-1
N_x	$2EIR\frac{2k^4}{\gamma^3}\cos kx$	-1	$1+\gamma$	1	$1-\gamma$
Q_x'	$-\frac{2EIk^3}{\gamma}\sin kx$	$\gamma+2$	2	$\gamma-2$	2
u	$2EI\frac{2k^3}{h\gamma^3}\sin kx$	-1	$1+\gamma$	1	$1-\gamma$
w	$2\cos kx$	1	0	1	0

Table 16.3 Stress resultants and displacements due to complementary function (Odd)

$F = \bar{R}$ $[(aB_1 - bB_2)\cos\beta_1\phi \sinh \alpha_1\phi - (aB_2 + bB_1)\sin\beta_1\phi \cosh \alpha_1\phi$
$+(cB_3 - dB_4)\cos\beta_2\phi \sinh \alpha_2\phi - (cB_4 + dB_3)\sin\beta_2\phi \cosh \alpha_2\phi]$

F	$\bar{R}$	B_1	B_2	B_3	B_4
Q_ϕ	$\frac{2EIk^3}{(\gamma)^{3/2}}\cos kx$	$m_1 - n_1$	$m_1 + n_1$	$-m_2 - n_2$	$m_2 - n_2$
Q_ϕ'	$\frac{2EIk^3}{\gamma^{3/2}}\cos kx$	$m_1(1-\gamma) - n_1$	$m_1 + n_1(1-\gamma)$	$-m_2(1+\gamma) - n_2$	$m_2 - n_2(1+\gamma)$
$N_{x\phi}$	$2EIR\frac{2k^4}{\gamma^{3/2}}\sin kx$	$-n_1$	m_1	n_2	$-m_2$
$M_{x\phi}$	$\frac{-2EIK}{R}\sin kx$	α_1	β_1	α_2	β_2
v	$-2IR\frac{2k^3}{h\gamma^{7/2}}\cos kx$	$m_1 + n_1(1-\gamma)$	$n_1 - m_1(1-\gamma)$	$-m_2 + n_2(1+\gamma)$	$-n_2 - m_2(1+\gamma)$
θ	$\cos kx$	$-\frac{2\alpha_1}{R}+\frac{(\bar{R}B_1)v}{R}$	$-\frac{2\beta_1}{R}+\frac{(\bar{R}B_2)v}{R}$	$-\frac{2\alpha_2}{R}+\frac{(\bar{R}B_3)v}{R}$	$-\frac{2\beta_2}{R}+\frac{(\bar{R}B_4)v}{R}$

Note: In the Table 16.3, θ is the rotation of the tangent.

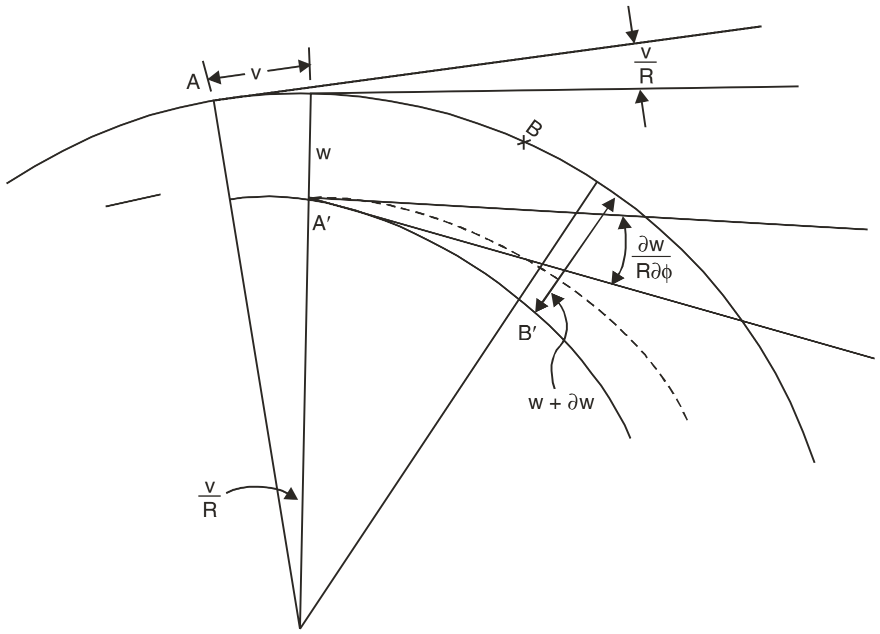


Fig. 16.8

Referring to Fig. 16.8,

Due to differential radial displacement, tangent at A will rotate by $\dfrac{\partial w}{R\partial\phi}$

Because of circumferential displacement v, at point A, tangent rotates by additional angle v/R,

$$\therefore \qquad \theta = \frac{\partial w}{R\partial\phi} + \frac{v}{R} \qquad \text{...eqn. 16.26}$$

16.10 EDGE CONDITIONS

To determine the arbitrary constants in the expression for displacement w, we have to make use of the edge conditions. The edge conditions in a shell naturally depend upon the method of supporting the edges. At each edge $\phi = \phi_k$, one has to look for any four of the moments, forces and displacements at edges, namely any four of M_ϕ, N_ϕ, Q'_ϕ, $N_{\phi x}$, u, v, w and θ. Such conditions for some of the common support conditions are presented below:

1. Edge Unsupported

In this case, since the shell edge is free, the forces and moments at the edges $\phi = \phi_k$, will be zero. Thus,

$$M_\phi = 0,\ N_\phi = 0,\ Q'_\phi = 0 \text{ and } N_{\phi x} = 0. \qquad \text{...eqn. 16.27}$$

2. Edge Supported on Unyielding Wall

In such case, the wall restricts vertical and horizontal displacement at the shell but allows rotation. Horizontal and vertical components are found by taking components of v and w. They are

$$w \sin Q_k - v \cos \phi_k \qquad \text{and } w \cos \phi_k + v \sin \phi_k.$$

Hence, the boundary conditions are,

$$w \sin Q_k - v \cos \phi_k = 0, \qquad w \cos \phi_k + v \sin \phi_k = 0.$$

Since, edge rotation is permitted, the other two edge conditions are,

$$M_\phi = 0 \text{ and } N_{x\phi} = 0. \qquad \text{...eqn. 16.28}$$

3. Edge Rigidly Held (Fixed Edge)

This is not so common case. Since, it is fixed edge conditions, all displacements are zero at the edge. Thus,

$$u_{\phi k} = 0, v_{\phi k} = 0, w_{\phi k} = 0, \theta_{\phi k} = 0. \qquad \text{...eqn. 16.29}$$

4. Edge Provided with Edge Beam

This is a common case. For **outer edge** of a multiple shell or the edges of single barrel shell, the following boundary conditions may be assumed:

(*a*) $\theta = 0$, *i.e.* no rotation of shell edge owing to relatively great stiffness of edge beam.

(*b*) $R_H = 0$, *i.e.* edge beam is incapable of withstanding lateral thrust. Since, N_ϕ and Q'_ϕ are the forces giving horizontal component of forces,

$R_H = N_\phi \cos\phi_k - Q'_\phi \cdot \sin\phi_k = 0$

(*c*) The longitudinal displacements of shell and edge beam should be the same.

i.e. $v_S = v_B$.

(*d*) The vertical displacement of shell and edge beam should be the same.

i.e. $w \cos\phi_k - v \sin\phi_k = w_B$ ...eqn. 16.30

For **inner edge** of a multiple shell, the following boundary conditions may be assumed:

(*a*) $v = 0$

(*b*) Displacement in horizontal direction = 0
i.e. $w \sin\phi_k + v \cos\phi_k = 0$.

(*c*) $u_S = u_B$, and

(*d*) $w_s = w \cos \phi_k - v \sin \phi_k = w_B$. ...16.31

16.11 EDGE BEAM THEORY

The relationship between edge beam forces and displacements with respect to the shell edges forces should be found so as to apply necessary boundary conditions. Edge beam theory deal with this aspect of shell analysis.

Shell exerts following forces on the edge beam: N_ϕ, $N_{\phi x}$, Q_ϕ, M_ϕ and $M_{x\phi}$.

Equal and opposite stress resultants develop on edge beam. Let the connection be at the corner of the edge beam. As a result of these forces from the shell edge beam is subjected to the following forces:

(*a*) F – longitudinal force,

(*b*) S – vertical shear and

(*c*) M_x – bending moment.

Consider the equilibrium of unit length of edge beam as shown in Fig. 16.9.

Σ Forces in x-direction = 0, gives,

$$-F + F + \frac{\partial F}{\partial x} \times 1 - N_{\phi x} = 0$$

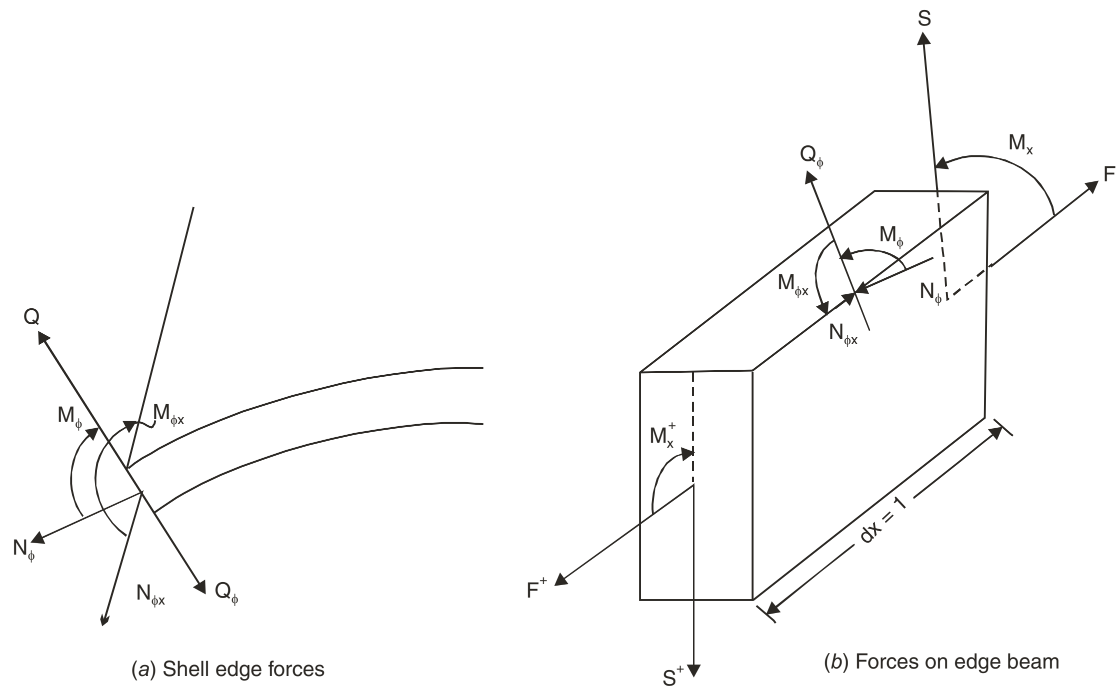


Fig. 16.9

i.e.
$$\frac{\partial F}{\partial x} = N_{\phi x}$$

$\therefore$
$$F = \int N_{\phi x}\, dx$$
$$= \int [N_{\phi x}] \sin kx\, dx$$
$$= -\frac{1}{k}\left[N_{\phi x}\right] \cos kx \qquad \ldots 16.32(a)$$

where $[N_{\phi x}] \sin kx = N_{\phi x}$.

Similarly for all stress resultants, the magnitude excluding sin *kx* or cos *kx* associated with will be marked as a bracked quantity and edge beam theory derived.

Σ Forces in vertical direction = 0, gives,
$$\frac{\partial S}{\partial x} - Q_\phi \cos\phi_k - N_\phi \sin\phi_k + W' = 0$$

where W' is weight of edge beam per unit length.

$\therefore$
$$S = \int \left(Q_\phi \cos\phi_k + N_\phi \sin\phi_k - W'\right) dx$$
$$= \frac{1}{k}\left\{\left[Q_\phi\right]\cos\phi_k + \left[N_\phi\right]\sin\phi_k - [W']\right\} \sin kx$$

Σ Moments in x-direction = 0, gives,

$$\frac{\partial M}{\partial x} + N_{x\phi} a_1 - M_{x\phi} \cos\phi_k - S \times 1 = 0$$

$$M = -\int \left\{ \left[N_{x\phi} \right] a_1 - \left[M_{x\phi} \right] \cos\phi_k - [S] \right\} \sin kx$$

$$= \frac{1}{k} \left\{ \left[N_{x\phi} \right] a_1 - \left[M_{x\phi} \right] \cos\phi_k - [S] \right\} \cos kx$$

$$= \frac{1}{k} \left\{ \left[N_{x\phi} \right] a_1 - \left[M_{x\phi} \right] \cos\phi_k - \left[Q_\phi \right] \frac{\cos\phi_k}{k} - \left[N_\phi \right] \frac{\sin\phi_k}{k} + \frac{[W']}{k} \right\} \cos kx$$

Stress at junction,

$$E \frac{\partial u_B}{\partial x} = \frac{F}{A} - \frac{M}{I_B} a_1$$

$$\therefore \qquad u_B = \frac{1}{E} \int \left(\frac{F}{A} - \frac{M}{I_B} a_1 \right) dx$$

$$= \frac{1}{E} \int \left[-\frac{1}{Ak} \left[N_{x\phi} \right] - \frac{a_1}{I_B k} \left\{ \left[N_{x\phi} \right] a_1 - \left[M_{x\phi} \right] \cos\phi_k - \frac{Q_\phi}{k} \cos\phi_k - \frac{\left[N_\phi \right]}{k} \sin\phi_k + \frac{[W']}{k} \right\} \right] \cos kx \, dx$$

$$= \left[\left[-\frac{1}{AEk^2} \left[N_{x\phi} \right] \right] - \frac{a_1}{I_B k^2} \left\{ \left[N_{x\phi} \right] a_1 - \left[M_{x\phi} \right] \cos\phi_k - \frac{\left[Q_\phi \right]}{k} \cos\phi_k - \frac{\left[N_\phi \right]}{k} \sin\phi_k + \frac{[W']}{k} \right\} \right] \sin kx$$

Vertical deflection of edge beam is given by,

$$EI_B \frac{\partial^2 w_B}{\partial x^2} = M$$

$$= \frac{1}{k} \left\{ \left[N_{x\phi} \right] a_1 + \left[M_{x\phi} \right] \cos\phi_k - \frac{\left[Q_\phi \right]}{k} \cos\phi_k - \frac{\left[N_\phi \right]}{k} \sin\phi_k + \frac{[W']}{k} \right\} \cos kx.$$

$$\therefore \qquad w_B = -\frac{1}{I_B k^3} \left\{ \left[N_{x\phi} \right] a_1 + \left[M_{x\phi} \right] \cos\phi_k - \frac{Q_\phi}{k} \cos\phi_k - \frac{\left[N_\phi \right]}{k} \sin\phi_k + \frac{[W']}{k} \right\} \cos kx$$

16.12 SUMMARY OF CALCULATIONS

1. Select the overall dimensions of the shell using I.S. Recommendations. Calculate $k = \pi/L$.
2. Find membrane or particular solution.
3. Calculate:

(*a*) $\rho = \left(\frac{hR^6 k^4}{I} \right)^{1/8} = \left(\frac{12 R^6 k^4}{h^2} \right)^{1/8}$

(*b*) $\gamma = \dfrac{k^2 R^2}{\rho^2} \sqrt{2}$

(*c*) Find

$$\alpha_1 = \frac{\rho}{\sqrt[4]{2}} \left[\frac{\sqrt{(1+\gamma)^2 + 1} + 1 + \gamma}{2} \right]^{1/2}$$

$$\alpha_2 = \frac{\rho}{\sqrt[4]{2}} \left[\frac{\sqrt{(1-\gamma)^2 + 1} - (1-\gamma)}{2} \right]^{1/2}$$

$$\beta_1 = \frac{\rho}{\sqrt[4]{2}} \left[\frac{\sqrt{(1+\gamma)^2 + 1} - (1+\gamma)}{2} \right]^{1/2}$$

and

$$\beta_2 = \frac{\rho}{\sqrt[4]{2}} \left[\frac{\sqrt{(1-\gamma)^2 + 1} - (1-\gamma)}{2} \right]^{1/2}$$

(*d*) Find

$$m_1 = \frac{\sqrt[4]{2}}{\rho} \alpha_1$$

$$m_2 = \frac{\sqrt[4]{2}}{\rho} \alpha_2$$

$$n_1 = \frac{\sqrt[4]{2}}{\rho} \beta_1$$

$$n_2 = \frac{\sqrt[4]{2}}{\rho} \beta_2$$

(*e*) Select the boundary conditions at the edge $\phi = \phi_k$. Each equation can be formed by collecting coefficients of *a, b, c* and *d* and writing the right hand side of equation.

(*f*) Solve simultaneous equations to get arbitrary constants *a, b, c* and *d.*

(*g*) Using Table 16.2 and 16.3 find the stress resultants.

Since, the calculations are lengthy bending analysis is not meant for hand calculation. It is suggested that going through above steps one should develop analysis package and use it.

16.13 STATICAL CHECKS

For confirming the validity of the results obtained, the following statical checks should be applied.

1. **Sum of the horizontal forces in *x*-direction should be zero.**
   To apply this check, N_x should be found at mid point of various equal segments (say 5° or 10°).

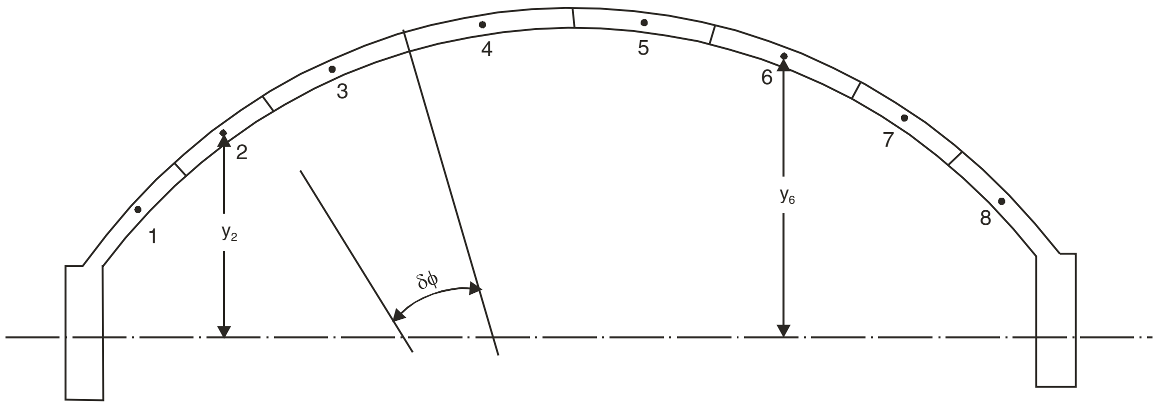


Fig. 16.10 Shell arch divided into 8 equal segments

Let these values be $N_{x_1}, N_{x_2}, ..., N_{x_L}$ as shown in Fig. 16.10. Apply Simpson's rule to sum up there forces. Thus

$$\Sigma N_x = \frac{R\delta_\phi}{3}\left[N_{x_1} + 4N_{x_2} + 2N_{x_3} + 4N_{x_4} + ...N_{x_L}\right]$$

where δ_ϕ is the angular length of the segment and N_{xL} is the last ordinate. Then the statical check is, if F is the sum of longitudinal forces in edge beams, then error $= \frac{\Sigma N_x - F}{\Sigma N_x} \times 100$.

This error should be within the limit.

2. **Sum of vertical components of $N_{x\phi}$ along the rim of end frame** (*i.e.* at $x = L/2$) **should be equal to half the total vertical load on the shell.**
Now, vertical component of $N_{x\phi}$

$$= N_{x\phi} \sin \phi_k$$

For finding the total vertical component due to $N_{x\phi}$ at $x = L/2$, find the values at mid point of various equal segments of length $R\delta\phi$. Then from Simpson's rule

$$\Sigma N_{x\phi} \sin\phi = R \times 2\phi_x \times \frac{L}{2} \times g.$$

One can make use of symmetry and consider only half the shell arch.

3. **Check for Longitudinal Moments**
About any horizontal axis $y - y$, in the cross section, sum of all moments must be equal to statical moment. This is applied at mid span ($x = 0$).
Referring to Fig. 16.10, various internal moments are
(*a*) Moment M_x in the shell.

$$\Sigma M_x = \frac{1}{3}R\delta\phi\left[M_{x_1} + 4M_{x2} + 2M_{x_3} + 4M_{x_4} + ... + M_{x_L}\right]$$

(*b*) Moment due to N_x forces

$$= \frac{R\delta_\phi}{3}\left[y_1 N_{x_1} + 4y_2 N_{x_2} + 2y_3 N_{x_3} + 4y_4 N_{x_4} + ... + y_L N_{xL}\right]$$

where y_1, y_2 ... are the lever arms of N_x forces in the middle of segments.

$$= R(\cos\phi - \cos\phi_k) + h/2$$

(*c*) Edge Beam Moment = $2M_1$

(*d*) Statical moment is due to sinusoidal loading on the shell and that on the edge beam:

At mid span.

$$= \left[\frac{L^2}{\pi^2}(R \times 2\phi_k g') + w'\frac{L^2}{\pi^2}\right]$$

The error is, $\dfrac{\text{Sum of moments } (a), (b), (c) - \text{moment}(d)}{\text{moment}(d)}$. It should be within the acceptable limit.

16.14 LONG AND SHORT SHELLS

In the previous discussion it has been said that beam theory and Schorer theory are applicable only for long shells. D.K.J. theory is applicable for long as well as short cylindrical shells. However, the terms long and short shells have no precise meaning. It is difficult to exactly demarcate between them. Several criteria have been proposed from time to time by different authors. Some of them are presented in this article.

1. Basis of Applicability of Beam Theory

Beam theory is the simple method for the analysis of cylindrical shell. It is hoped on the assumption that along the depth of beam stress variation is linear. After studying the stress distribution across the depth by more precise theories it has been suggested that beam theory is applicable if,

(*i*) $L/R \geq 5$ for shells without edge beams

(*ii*) $L/R \geq 3$ for shells with edge beams

Hence a shell is called long shell if $L/R \geq 5$, if it is without edge beam and if $L/R \geq 3$, if it is with edge beam.

2. Basis of Ignoring M_x, Q_x, M_{xy}

There are theories like Schorer theory in which the terms M_x, Q_x and M_{xy} in a shell element are taken as zero. Such theories are applicable only if, $L/R \geq \pi$. Hence, on the basis of ignoring M_x, M_{xy}, Q_x shells may be called as long shell if $L/R \geq \pi$.

3. Extent of Travel of Edge Disturbances

Disturbance emanating from the edges of shell may or may not penetrate beyond the crown. In long shells they do not penetrate crown *i.e.* actual forces at crown are the membrane forces only. On this basis ASCE classifies shell as long if $L/R > 16$.

4. Aas Jacobsen's Classification

Aas Jacobsen was perhaps the first to classify shells into long, intermediate and short. According to him,

the shell is long, if $\rho = 4$ to 7
intermediate, if $\rho = 7$ to 10
short, if $\rho = 10$ to 20.

QUESTIONS

1. Draw a typical cylindrical shell element and indicate various membrane forces, transverse shears and moments. Give the relations among forces and moments on positive faces and negative faces.
2. Derive the six equations of equilibrium of a shell element subject to bending.
3. Enumerate the basic assumptions made in the analysis of cylindrical shells in the following theories:
   (*i*) D. K. J. theory
   (*ii*) Schorer's theory.
   Comment on the above two theories.
4. Discuss edge conditions to be used in the following cases of cylindrical shell analysis:
   (*i*) Edge unsupported
   (*ii*) Edge supported on unyielding wall
   (*iii*) Fixed edge
   (*iv*) Inner edge of multiple shell.

Chapter

17

Analysis of Symmetrically Loaded Cylindrical Water Tanks and Pipes

Circular pipes and water tanks are usually subjected to symmetrical load and they are commonly used structure. In this chapter, the analysis of these structures is presented.

17.1 FORCES ON AN ELEMENT

Figure 17.1 shows a typical cylindrical pipe/water tank and Figure 17.2 shows an element in which

x – longitudinal direction

y or ϕ – tangential direction and

z – radial inward direction.

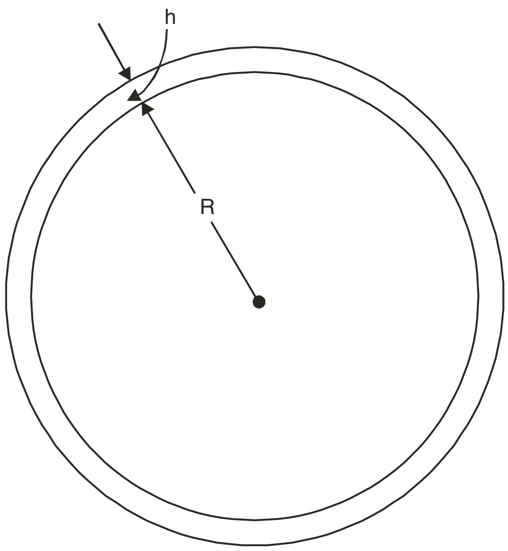


Fig. 17.1 A Cylindrical Pipe/Water tank

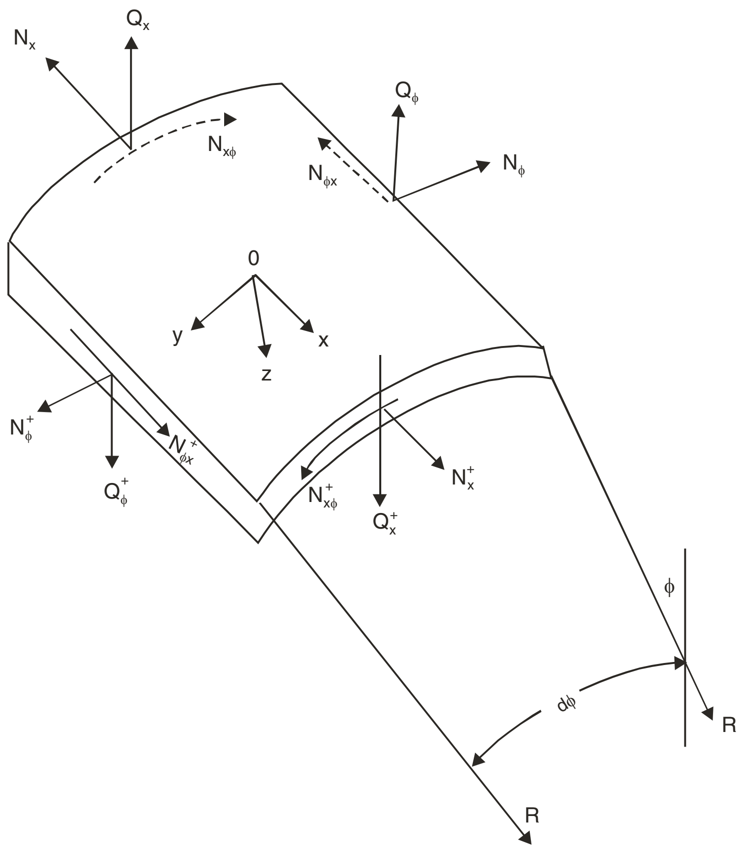


Fig. 17.2 Typical element

Due to symmetry,

(*i*) Shearing forces and twisting moments are zero *i.e.*

$$N_{x\phi} = N_{\phi x} = 0$$
$$Q_\phi = 0$$
$$M_{x\phi} = M_{\phi x} = 0.$$

(*ii*) There is no variation of forces and moments with respect to ϕ, *i.e.*

$$N_\phi \text{ – Constant}$$
$$M_\phi \text{ – Constant.}$$

Thus, there is variation of forces with respect to x only *i.e.*

$$N_x^+ = N_x + \frac{\partial N_x}{\partial x} dx$$

$$Q_x^+ = Q_x + \frac{\partial Q_x}{\partial x} dx \text{ and}$$

$$M_x^+ = M_x + \frac{\partial M_x}{\partial x} dx.$$

17.2 EQUATIONS OF EQUILIBRIUM

Let X and Z be component of load in x and z direction respectively.

(i) Σ Forces in x-direction = 0, gives

$$\left(N_x + \frac{\partial N_x}{\partial x}dx\right)R\,d\phi - N_x R d\phi + X dx R d\phi = 0$$

i.e.
$$\frac{\partial N_x}{\partial x} + X = 0. \qquad \text{...eqn. 17.1}$$

(ii) Σ Forces in z-direction = 0, gives

$$\left(Q_x + \frac{\partial Q_x}{\partial x}dx\right)Rd\phi - Q_x Rd\phi + N_\phi dx \sin\frac{d\phi}{2} + N_\phi dx \sin\frac{d\phi}{2} + Zdx\,Rd\phi = 0.$$

Since, ϕ is a small angle $\sin\frac{d\phi}{2} = \frac{d\phi}{2}$ and hence, the equilibrium equation reduces to

$$\frac{\partial Q_x}{\partial x}dx\,Rd\phi + N_\phi dx\,d\phi + Z\,R\,dxd\phi = 0$$

i.e.
$$\frac{\partial Q_x}{\partial x} + \frac{N_\phi}{R} + Z = 0 \qquad \text{...eqn. 17.2}$$

(iii) Σ Moments in x-direction = 0, gives

$$\left(M_x + \frac{\partial M_x}{\partial x}dx\right)Rd\phi - M_x Rd\phi - Q_x Rd\phi\frac{dx}{2} - \left(Q_x + \frac{\partial Q_x}{\partial x}dx\right)Rd\phi \cdot \frac{dx}{2} = 0$$

After neglecting small quantity of higher order,

$$\frac{\partial M_x}{\partial x}Rdxd\phi - Q_x Rd\phi dx = 0$$

i.e.
$$\frac{\partial M_x}{\partial x} - Q_x = 0 \qquad \text{...eqn. 17.3}$$

17.3 STRESS RESULTANTS

First equation gives N_x value directly and N_x term is not appearing in any of the subsequent equations. Hence, N_x can be determined independently. Due to other load components N_x is not at all affected.

Now consider the equations 2 and 3. There are three unknowns namely, Q_x, N_ϕ and M_x. As the number of unknowns are more than number of equations, compatibility of displacements should be considered.

Let u, v and w be the displacement components in x, y and z-directions respectively. Due to symmetry v is zero. Thus, only u and w exist.

Figure 17.3 shows the deformed position of the middle surface.

Now,
$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \text{...eqn. 17.4}$$

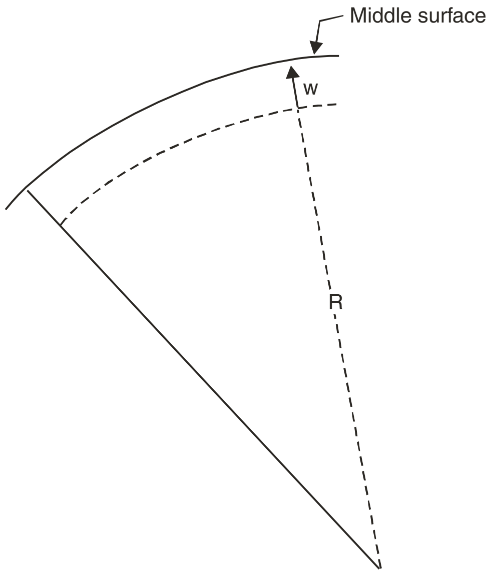


Fig. 17.3 Deformed position of middle surface

$$\varepsilon_\phi = \frac{\text{Final length in } \phi \text{ direction} - \text{Original length}}{\text{Original length}}$$

$$= \frac{(R-w)\,d\phi - R d\phi}{R\,d\phi} = -\frac{w}{R} \qquad \text{...eqn. 17.5}$$

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\mu\sigma_\phi}{E}, \text{ and}$$

$$\varepsilon_\phi = -\frac{\mu\sigma_x}{E} + \frac{\sigma_\phi}{E}$$

$\therefore$ $$\varepsilon_x + \mu\varepsilon_\phi = \frac{\sigma_x}{E} - \frac{\mu^2\sigma_x}{E}$$

$\therefore$ $$\sigma_x = \frac{E}{1-\mu^2}\left(\varepsilon_x + \mu\varepsilon_\phi\right)$$

$\therefore$ $$N_x = h\sigma_x = \frac{Eh}{1-\mu^2}\left(\varepsilon_x + \mu\varepsilon_\phi\right), \text{ where } h \text{ is the thickness.}$$

Similarly, $$N_\phi = \frac{Eh}{1-\mu^2}\left(\varepsilon_\phi + \mu\varepsilon_x\right)$$

Thus, $$N_x = \frac{Eh}{1-\mu^2}\left(\varepsilon_x + \mu\varepsilon_\phi\right) \qquad \text{...eqn. 17.6}(a)$$

and $$N_\phi = \frac{Eh}{1-\mu^2}\left(\varepsilon_\phi + \mu\varepsilon_x\right) \quad \text{...eqn. 17.6(b)}$$

For the loading Y and Z only, $N_x = 0$.

$\therefore$ $$\varepsilon_x = -\mu\varepsilon_\phi \quad \text{...eqn. 17.7}$$

Substituting eqn. 7 in eqn. 17.6(*b*), we get,

$$N_\phi = \frac{Eh}{1-\mu^2}\left(\varepsilon_\phi - \mu^2\varepsilon_\phi\right)$$

$$= Eh\varepsilon_\phi$$

$\therefore$ $$N_\phi = -\frac{Ehw}{R} \quad \text{...eqn. 17.8}$$

Due to symmetry, there is no change in curvature in the circumferential direction. The curvature in x-direction is equal to $-\frac{\partial^2 w}{\partial x^2}$.

$\therefore$ $$M_x = -D\frac{\partial^2 w}{\partial x^2} \text{ and} \quad \text{...eqn. 17.9(a)}$$

$$M_\phi = -D\mu\frac{\partial^2 w}{\partial x^2}$$

$$= \mu M_x \quad \text{...eqn. 17.9(b)}$$

where $$D = \frac{Eh^3}{12\left(1-\mu^2\right)}$$

From equation of equilibrium 17.3,

$$Q_x = \frac{\partial M_x}{\partial x} = -D\frac{\partial^2 w}{\partial x^3}$$

From equation 17.2,

$$-D\frac{\partial^4 w}{\partial x^4} - \frac{Ehw}{R} \times \frac{1}{R} + Z = 0.$$

$\therefore$ $$-\frac{\partial^4 w}{\partial x^4} - \frac{Eh}{R^2 D}w + \frac{Z}{D} = 0$$

i.e. $$\frac{\partial^4 w}{\partial x^4} + \frac{Eh}{R^2 D}w = \frac{Z}{D}$$

i.e. $$\frac{\partial^4 w}{\partial x^4} + 4\beta^4 w = \frac{Z}{D} \quad \text{...eqn. 17.10}$$

where $$4\beta^4 = \frac{Eh}{R^2 D}$$

i.e. $$\beta^4 = \frac{Eh}{4R^2 D} = \frac{Eh}{4R^2 \frac{Eh^3}{12(1-\mu^2)}} = \frac{3(1-\mu^2)}{R^2 h^2}$$

Equation 17.10 is to be solved to get particular integral and complementary solution. Particular integral depends upon the nature of loading and complementary solution depends upon the boundary conditions. Thus, the total solution

$$w = W_1 + W_2$$

where w_1 — particular integral

and w_2 — complementary solution

Complementary solution is given by

$$\frac{\partial^4 w_2}{\partial x^4} + 4\beta^4 w_2 = 0$$

$$(\partial^4 + 4\beta^4) w_2 = 0$$

$\therefore$ $$\delta^4 = -4\beta^4 = 4\beta^4 [\cos(2n\pi + \pi) + i \sin(2n\pi + \pi)]$$

$\therefore$ $$\delta = \sqrt{2}\,\beta \left[\cos\frac{2n\pi + \pi}{4} + i \sin\frac{2n\pi + \pi}{4} \right]$$

where $n = 0, 1, 2$ and 3.

i.e. $$\delta_1 = \sqrt{2}\,\beta \left[\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} \right] = \beta[1 + i]$$

$$\delta_2 = \sqrt{2}\,\beta \left[-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} \right] = \beta[-1 + i]$$

$$\delta_3 = \sqrt{2}\,\beta \left[-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}} \right] = \beta[-1 - i]$$

$$\delta_4 = \sqrt{2}\,\beta \left[\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}} \right] = \beta[1 - i]$$

$\therefore$ $$w_2 = C_1' e^{(1+i)\beta x} + C_2' e^{(1-i)\beta x} + C_3 e^{(-1-i)\beta x} + C_4 e^{(-1+i)\beta x}$$

$$= C_1' e^{\beta x} [\cos\beta x + i \sin\beta x] + C_2' e^{\beta x} [\cos\beta x - i \sin\beta x]$$

$$+ C_3' e^{-\beta x} [\cos\beta x - i \sin\beta x] + C_4' e^{-\beta x} [\cos\beta x + i \sin\beta x]$$

$$= e^{\beta x}\left(C_1' + C_2\right)\cos\beta x + e^{\beta x}\left(C_1' - C_2'\right)i\sin\beta x$$

$$+ e^{-\beta x}\left(C_3' + C_4'\right)\cos\beta x + e^{-\beta x}\left(C_4 - C_3\right)i\sin\beta x$$

It is to be noted that C_1', C_2', C_3' and C_4' are complex numbers. Since, w is real, it follows that $C_1' + C_2'$, $i(C_1' - C_2')$, $C_3' + C_4'$ and $i(C_3' - C_4')$ should be real. It means C_1 and C_2 are conjugate terms and C_3 and C_4 are also conjugate. Hence, let

$$C_1 = C_1' + C_2',\ C_2 = i(C_1' - C_2'),\ C_3 = (C_3' + C_4')$$

and $C_4 = i(C_3' - C_4')$, where C_1, C_2, C_3 and C_4 are real terms.

$$\therefore \quad w_2 = C_1 e^{\beta x}\cos\beta x + C_2 e^{\beta x}\sin\beta x + C_3 e^{-\beta x}\cos\beta x + C_4 e^{-\beta x}\sin\beta x$$

$$= e^{\beta x}\left(C_1\cos\beta x + C_2\sin\beta x\right) + e^{-\beta x}\left(C_3\cos\beta x + C_4\sin\beta x\right) \qquad \text{...eqn. 17.11}$$

Example 17.1. Analyse a water tank of radius R and depth 'd', if its one edge is fixed at base slab and top is free. Assume the thickness of wall uniform.

Solution. Figure 17.4 shows a typical water tank. Let the coordinates be selected as shown in the figure.

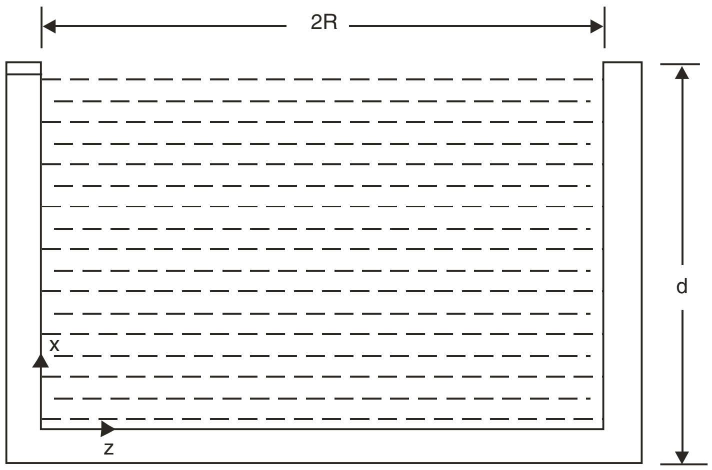


Fig. 17.4 A Typical water tank

In this case at x, the load component is

$$Z = -\gamma(d - x)$$

where γ—unit weight of water.

$$\therefore \quad \frac{d^4 w}{dx^4} + 4\beta^4 w = -\frac{\gamma(d - x)}{D}$$

Particular Solution:

$$\left(\delta^4 + 4\beta^4\right)w_1 = \frac{-\gamma(d - x)}{D}$$

$$4\beta^4\left(1+\frac{\delta^4}{4\beta^4}\right)w_1 = -\frac{\gamma(d-x)}{D}$$

$$w_1 = \frac{-\gamma(d-x)}{4\beta^4 D}\left(1+\frac{\delta^4}{4\beta^4}\right)^{-1}$$

$$= -\frac{\gamma(d-x)}{4\beta^4 D}\left[1+\frac{1}{2}\frac{\delta^4}{4\beta^4}-...\right]$$

$$= -\frac{\gamma(d-x)}{4\beta^4 D}$$

$$= -\frac{\gamma(d-x)}{\frac{Eh}{R^2 D}D}$$

∴ $$w_1 = -\gamma(d-x)\frac{R^2}{Eh}$$ ...eqn. 17.12

The same result is obtained if we consider membrane solution, as shown below:

$$Z = -\gamma(d-x)$$

Referring to Fig. 17.5

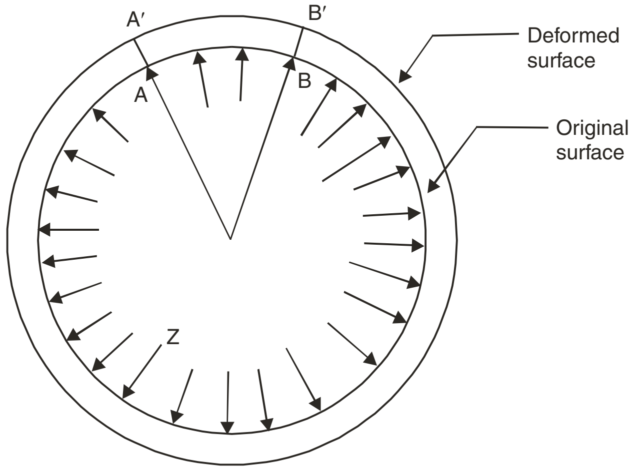


Fig. 17.5 Cross section of a water tank

$$\text{Hoop stress} = \gamma(d-x)\frac{R}{h}$$

∴ $$\text{Circumferential strain} = \frac{\gamma(d-x)R}{Eh}$$

i.e. $$\frac{A'B'-AB}{AB}=\gamma\frac{(d-x)R}{Eh}$$

i.e. $$\frac{(R-w)d\phi-Rd\phi}{Rd\phi}=\gamma\frac{(d-x)R}{Eh}$$

i.e. $$-\frac{w}{R}=\frac{\gamma(d-x)R}{Eh}$$

or $$w=\frac{-\gamma(d-x)R^2}{Eh}$$

Thus, the particular solution and the membrane solutions are the same. Thus, here also particular solution can replace membrane solution. The complementary solution which depends on boundary conditions may be looked as edge perturbations from boundaries.

Total solution,

$$w = w_1 + w_2$$

$$= e^{\beta x}(C_1 \cos \beta x + C_2 \sin \beta x) + e^{-\beta x}(C_3 \cos \beta x + C_4 \sin \beta x)-\frac{\gamma(d-x)R^2}{Eh} \quad \text{...eqn. 17.13}$$

In any stable structure disturbances at one end must go on reducing at the other end. w can go on reducing with increase in x if and only if C_1 and C_2 are zero.

Hence,

$$w=e^{-\beta x}\left(C_3\cos\beta x+C_4\sin\beta x\right)-\frac{\gamma(d-x)R^2}{Eh}$$

C_3 and C_4 are to be determined from the boundary conditions. In this case, the boundary conditions available are,

$$w_{x=0}=0 \quad \text{...(1)}$$

$$\left.\frac{\partial w}{\partial x}\right|_{x=0}=0 \quad \text{...(2)}$$

From B.C. (*i*), we get,

$$0=C_3-\frac{\gamma dR^2}{Eh}$$

or $$C_3=\frac{\gamma dR^2}{Eh}.$$

From B.C. (*ii*), we get,

$$\left.-\beta e^{-\beta x}\left(C_3\cos\beta x+C_4\sin\beta x\right)+e^{-\beta x}\left(-C_3\beta\sin\beta x+C_4\beta\cos\beta x\right)+\frac{\gamma R^2}{Eh}\right|_{x=0}=0.$$

i.e. $$-\beta C_3+C_4\beta+\frac{\gamma h^2}{Eh}=0$$

or,
$$C_4 = C_3 - \frac{\gamma R^2}{Eh\beta} = \frac{\gamma d R^2}{Eh} - \frac{\gamma R^2}{Eh\beta}$$

$$= \frac{\gamma R^2}{Eh}\left(d - \frac{1}{\beta}\right)$$

$\therefore$
$$w = -\frac{\gamma R^2}{Eh}\left[d - x - e^{-\beta x}\left\{d\cos\beta x + \left(d - \frac{1}{\beta}\right)\sin\beta x\right\}\right] \qquad \text{...eqn. 17.14}$$

$\therefore$
$$N_\phi = -\frac{Ehw}{R} = \gamma R\left[d - x - e^{-\beta x}\left\{d\cos\beta x + \left(d - \frac{1}{\beta}\right)\sin\beta x\right\}\right] \qquad \text{...eqn. 17.15}$$

Now,
$$\frac{\partial w}{\partial x} = \beta e^{-\beta x}\left(-C_3\cos\beta x - C_4\sin\beta x - C_3\sin\beta x + C_4\cos\beta x\right) + \frac{\gamma R^2}{Eh}$$

$\therefore$
$$\frac{\partial^2 w}{\partial x^2} = \beta^2 e^{-\beta x}\left[\left(C_3\cos\beta x + C_4\sin\beta x + C_3\sin\beta x - C_4\cos\beta x\right) + C_3\sin\beta x\right.$$
$$\left. - C_4\cos\beta x - C_3\cos\beta x - C_4\sin\beta x\right]$$
$$= 2\beta^2 e^{-\beta x}\left(C_3\sin\beta x - C_4\cos\beta x\right)$$
$$= 2\beta^2 e^{-\beta x}\frac{\gamma R^2}{Eh}\left[d\sin\beta x - \left(d - \frac{1}{\beta}\cos\beta x\right)\right]$$

From eqn. 17.9(a),
$$M_x = -D\frac{\partial^2 w}{\partial x^2}$$

$\therefore$
$$M_x = -2\beta^2\frac{\gamma R^2}{Eh}De^{-\beta x}\left[d\sin\beta x - \left(d - \frac{1}{\beta}\right)\cos\beta x\right] \qquad \text{...eqn. 17.16}$$

Convergence Study

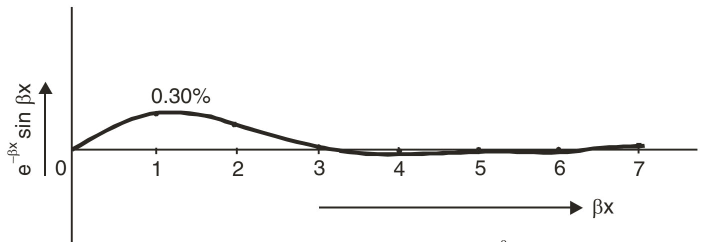
Table 17.1 shows the values of βx, $e^{-\beta x}\sin\beta x$ and $e^{-\beta x}\cos\beta x$. Figure 17.6 shows the variation of the functions $e^{-\beta x}\sin\beta x$ and $e^{-\beta x}\cos\beta x$ with respect to $e^{-\beta x}$. It may be observed that the two functions are converging fast. Hence, deflection and all stress resultants are converging fast. For example, for a water tank with $R = 5$ m, $d = 4.5$ m, $h = 150$ mm, if μ is taken zero,

$$\beta^4 = \frac{3\left(1-\mu^2\right)}{R^2h^2} = \frac{3}{5^2\,(0.15)^2} = 5.333$$

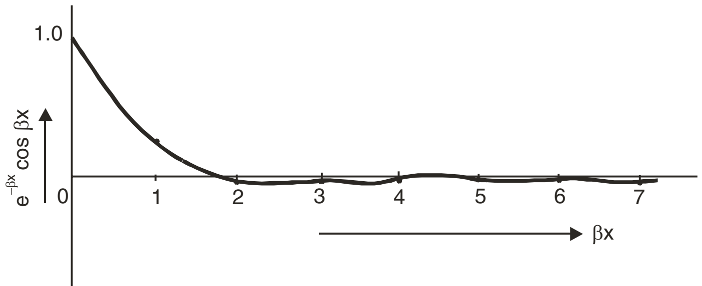
$\therefore$
$$\beta = 1.52$$

Table 17.1 Convergence study of the functions

βx	$e^{-\beta x}\sin\beta x$	$e^{-\beta x}\cos\beta x$
0	0	1.0
1	0.3096	0.1988
2	0.1231	–0.0563
3	0.0071	–0.0493
4	–0.0139	–0.0120
5	–0.0065	0.0019
6	–0.0007	0.0024
7	0.0006	0.0007



(*a*) Convergence of the function $e^{-\beta x}\sin\beta x$



(*b*) Convergence of the function $e^{-\beta x}\cos\beta x$

Fig. 17.6 Convergence of the functions

At $x = 0$, at edge fixed with slab,

$$e^{-\beta x} = 1,\ \sin\beta x = 0,\ \cos\beta x = 1$$

$$\therefore \qquad M_0 = \frac{-2\beta^2\gamma R^2}{Eh} D\left[-\left(d - \frac{1}{\beta}\right)\right]$$

$$= 2\beta^2\gamma R^2 \frac{Eh^3}{12(1-\mu^2)} \times \frac{1}{Eh}(d-\beta)$$

$$= 2\beta^2\gamma R^2 \frac{h^2}{12}(d-\beta)$$

Substituting $\beta = 1.52$, $\gamma = 9.8$ kN/m^3 and the other values we get,

$$M_0 = 2 \times 1.52^2 \times 9.8 \times 5.0^2 \frac{(0.15)^2}{12}\left(4.5 - \frac{1}{1.52}\right)$$

$$= 8.156 \text{ kN-m}$$

At the free edge $x = 4.5$ m, since

$$\frac{D}{Eh} = \frac{Eh^3}{12(1-0)} \times \frac{1}{Eh} = \frac{h^2}{12}$$

$\therefore$

$$M_0 = 2 \times 1.52^2 \times 9.8 \times 5^2 \times \frac{(0.15)^2}{12} \times e^{-1.52 \times 4.5}$$

$$\left[4.5 \sin 4.5 \times 1.52 - \left(4.5 - \frac{1}{1.52}\right)\cos 4.5 \times 1.52\right]$$

$$= 0.002 \text{ kN-m} \approx 0.$$

$\therefore$ The boundary condition at top edge do not influence moment at fixed edge.

At $\quad x = \dfrac{d}{3} = 1.5 \text{ m}, \ \therefore \beta x = 1.5 \times 1.52 = 2.28$

$$M_x = 2 \times 1.52^2 \times 9.8 \times 5^2 \times \frac{0.15^2}{12} e^{-2.28}\left[4.5 \sin 2.28 - \left(4.5 - \frac{1}{1.52}\right)\cos 2.28\right]$$

$$= 1.284 \text{ kN-m}$$

It may be noted that in R.C.C. water tank design, it is assumed that in top 2/3 depth vertical reinforcement may be only nominal reinforcement. The assumption is justified. In the design it is also assumed that the moment at base is due to the pressure shown in Fig. 17.7.

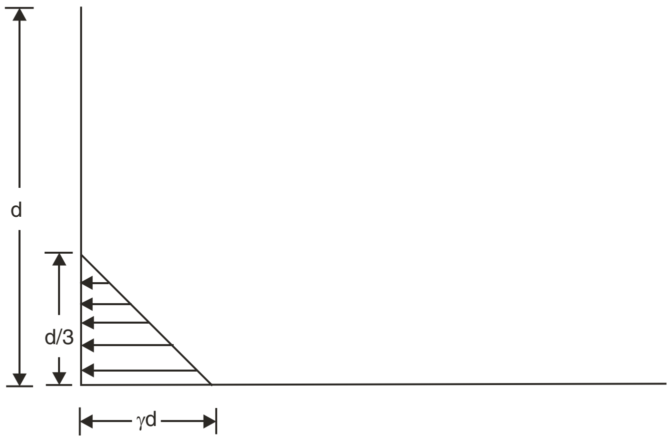


Fig. 17.7 Approximate estimation of moment M_x

i.e. $\quad M_0 = \dfrac{1}{2} \times \gamma d \times \dfrac{d}{3} \times \left(\dfrac{1}{3} \times \dfrac{d}{3}\right) = \dfrac{\gamma d^3}{54}$

Hence, in this case approximated moment is

$$M_0 = \frac{9.8 \times 4.5^3}{54} = 16.538 \text{ kN-m.}$$

This is almost double of actual value of 8.156 kN-m. Thus, in the approximate method moment is overestimated.

17.4 ANALYSIS OF LONG CIRCULAR PIPES

In the analysis of shells, particular solution can be replaced by membrane solution and complementary solution can be considered as edge perturbations. Hence, the two solutions can be found separately and superposed to get total solution. The perturbation at the edges is due to edge moment M_0 and edge transverse shear Q_0. Due to antisymmetry M_0 and Q_0 are uniformly distributed along the edges. The deflection function w due to M_0 and Q_0 may be found as explained below (Ref. Fig. 17.8).

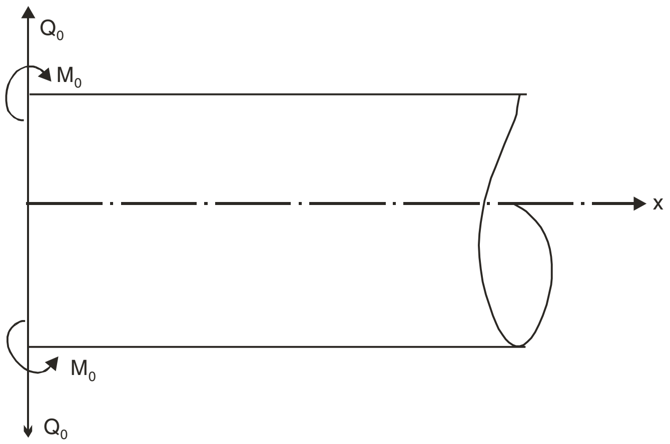


Fig. 17.8 Pipe subjected to uniform M_0 and Q_0

Since, the forces applied at the end $x = 0$ produce a local bending which dies out rapidly with x from the loaded end, we conclude that C_1 and C_2 in the general solution must vanish.

i.e. $$C_1 = C_2 = 0$$

$\therefore$ $$w = e^{-\beta x}(C_3 \cos \beta x + C_4 \sin\beta x)$$

The boundary conditions to be satisfied are:

$$M_x\Big|_{x=0} = -D\left(\frac{\partial^2 w}{\partial x^2}\right)_{x=0} = M_0 \qquad \text{...}(i)$$

and $$Q_x\Big|_{x=0} = \frac{\partial M_x}{\partial x}\bigg|_{x=0} = -D\left(\frac{\partial^3 w}{\partial x^3}\right)_{x=0} = Q_0 \qquad \text{...}(ii)$$

Now, $$\frac{\partial w}{\partial x} = \beta e^{-\beta x}\left(-C_3 \cos\beta x - C_4 \sin\beta x - C_3 \sin\beta x + C_4 \cos\beta x\right)$$

$$\frac{\partial^2 w}{\partial x^2} = \beta^2 e^{-\beta x}\left(C_3 \cos\beta x + C_4 \sin\beta x + C_3 \sin\beta x - C_4 \cos\beta x\right.$$

$$\left. + C_3 \sin\beta x - C_4 \cos\beta x - C_3 \cos\beta x - C_4 \sin\beta x\right)$$

$$= 2\beta^2 e^{-\beta x}\left(C_3 \sin\beta x - C_4 \cos\beta x\right)$$

$\therefore$
$$\frac{\partial^3 w}{\partial x^3} = 2\beta^3 e^{-\beta x}\left(-C_3 \sin\beta x + C_4 \cos\beta x + C_3 \cos\beta x + C_4 \sin\beta x\right)$$

From B.C. (*i*), we get,

$$-D2\beta^2\left(-C_4\right) = M_0 \quad \text{or} \quad C_4 = \frac{M_0}{2\beta^2 D}$$

From B.C. (*ii*), we get,

$$D2\beta^3\left(C_4 + C_3\right) = Q_0$$

i.e.
$$C_3 + C_4 = -\frac{Q_0}{2\beta^3 D}$$

i.e.
$$C_3 = -\frac{Q_0}{2\beta^3 D} - \frac{M_0}{2\beta^2 D}$$

or
$$C_3 = -\frac{1}{2\beta^3 D}\left(Q_0 + \beta M_0\right)$$

$\therefore$
$$w = \frac{e^{-\beta x}}{2\beta^3 D}\left[\beta M_0\left(\sin\beta x - \cos\beta x\right) - Q_0 \cos\beta x\right] \qquad \text{...eqn. 17.17}$$

The maximum deflection occurs at $x = 0$.

$$w_{\max} = w_0 = -\frac{1}{2\beta^3 D}\left[\beta M_0 + Q_0\right] \qquad \text{...eqn. 17.18}$$

The negative sign for this deflection is due to the fact that w is taken as positive towards the axis of the cylinder. The slope at $x = 0$ is given by

$$\left.\frac{\partial w}{\partial x}\right|_{x=0} = \beta\left(-C_3 + C_4\right)$$

$$= \beta\left[\left(Q_0 + \beta M_0\right)\frac{1}{2\beta^3 D} + \frac{M_0}{2\beta^2 D}\right]$$

$$= \frac{1}{2\beta^2 D}\left[Q_0 + 2\beta M_0\right] \qquad \text{...eqn. 17.19}$$

17.5 ANALYSIS OF WATER TANKS AS COMBINATION OF MEMBRANE SOLUTION AND EDGE PERTURBATIONS M_0, Q_0

The water tank may be analysed as the combination of the following two cases:

(*i*) Membrane analysis (Free edge at bottom also) and

(*ii*) Free edge subject to M_0 and Q_0 at bottom with the conditions that

$$\left.\begin{aligned} w &= 0 \text{ at } x = 0 \\ \text{and } \frac{\partial w}{\partial x} &= 0 \text{ at } x = 0 \end{aligned}\right\} \text{for fixed base}$$

$$\left.\begin{aligned} w &= 0 \text{ at } x = 0 \\ \text{and } M_x &= 0 \text{ at } x = 0 \end{aligned}\right\} \text{for hinged base.}$$

Analysis of Water Tank with Fixed Base:

Membrane displacement is given by

$$w_1 = \frac{-\gamma(d-x)}{4\beta^4 D}$$

and
$$w_2 = -\frac{1}{2\beta^3 D}\left[\beta M_0(\sin\beta x - \cos\beta x) - Q_0\cos\beta x\right]$$

$\therefore$
$$w_2\Big|_{x=0} = -\frac{1}{2\beta^3 D}(\beta M_0 + Q_0)$$

$\therefore$
$$\frac{\partial w_2}{\partial x}\Bigg|_{x=0} = \frac{1}{2\beta^2 D}(2\beta M_0 + Q_0)$$

Now, boundary conditions are at $x = 0$,

$$w = 0 \quad ...(1) \qquad \text{and} \qquad \frac{\partial w}{\partial x} = 0 \quad ...(2)$$

Noting that $w = w_1 + w_2$, from boundary condition (*i*), we get

$$-\frac{\gamma d}{4\beta^4 D} - \frac{1}{2\beta^3 D}(\beta M_0 + Q_0) = 0.$$

or
$$(\beta M_0 + Q_0) = \frac{-\gamma d}{2\beta} \qquad ...(iii)$$

From boundary condition (*ii*), we get,

$$\frac{\gamma}{4\beta^4 D} + \frac{1}{2\beta^2 D}(2\beta M_0 + Q_0) = 0$$

i.e.
$$2\beta M_0 + Q_0 = -\frac{\gamma}{2\beta^2} \qquad ...(iv)$$

From equation (*iii*) and (*iv*), we get,

$$\beta M_0 = \frac{\gamma d}{2\beta} - \frac{\gamma}{2\beta^2}$$

or
$$M_0 = \frac{\gamma}{2\beta^2}\left(d - \frac{1}{\beta}\right) \qquad ...(v)$$

From equation (*iii*),

$$Q_0 = -\frac{\gamma d}{2\beta} - \beta M_0$$

$$= -\frac{\gamma d}{2\beta} - \frac{\gamma}{2\beta}\left(d - \frac{1}{\beta}\right)$$

$$= \frac{\gamma}{\beta}\left[-d + \frac{1}{2\beta}\right]$$

$$\therefore \quad w = -\frac{\gamma(d-x)}{4\beta^4 D} + \frac{e^{-\beta x}}{2\beta^3 D}\left[\left(\frac{\gamma d}{2\beta} - \frac{\gamma}{2\beta^2}\right)(\sin\beta x - \cos\beta x) - \frac{\gamma}{\beta}\left(-d + \frac{1}{2\beta}\right)\cos\beta x\right]$$

$$= -\frac{\gamma(d-x)}{4\beta^4 D} + \frac{e^{-\beta x}}{2\beta^3 D}\left[\frac{\gamma d}{2\beta}\sin\beta x + \frac{\gamma d}{2\beta}\cos\beta x - \frac{\gamma}{2\beta^2}\sin\beta x\right]$$

$$= -\frac{\gamma(d-x)}{4\beta^4 D} + \frac{e^{-\beta x}}{2\beta^4 D}\left[\gamma d\sin\beta x + \gamma d\cos\beta x - \frac{\gamma}{\beta}\sin\beta x\right]$$

$$= -\frac{\gamma}{4\beta^4 D}\left[d - x - e^{-\beta x}\left\{\left(d - \frac{1}{\beta}\right)\sin\beta x + d\cos\beta x\right\}\right] \qquad ...\text{eqn. 17.20}$$

This is same as eqn. 17.14 obtained from first principle.

(*b*) **Tank with Hinged Base:**

The total solution is,

$$w = -\frac{\gamma(d-x)}{4\beta^4 D} + \frac{e^{-\beta x}}{2\beta^3 D}\left[\beta M_0(\sin\beta x - \cos\beta x) - Q_0\cos\beta x\right]$$

The boundary conditions are at $x = 0$,

$w = 0$ and $M_0 = 0$.

$\therefore$ we get
$$0 = -\frac{\gamma d}{4\beta^4 D} + \frac{1}{2\beta^3 D}\left[-Q_0\right] = 0.$$

$$\therefore \quad Q_0 = -\frac{\gamma d}{2\beta}$$

$$\therefore \qquad w = -\frac{\gamma(d-x)}{4\beta^4 D} + \frac{e^{-\beta x}}{2\beta^3 D}(-Q_0 \cos\beta x)$$

$$= -\frac{\gamma(d-x)}{4\beta^4 D} - \frac{e^{-\beta x}}{2\beta^3 D}\left(-\frac{\gamma d}{2\beta}\right)\cos\beta x$$

$$= -\frac{\gamma(d-x)}{4\beta^4 D} + \frac{\gamma d}{4\beta^4 D} e^{-\beta x}\cos\beta x$$

$$= -\frac{\gamma}{4\beta^4 D}\left[d - x - de^{-\beta x}\cos\beta x\right]$$

$$= -\frac{\gamma R^2}{Eh}\left[d - x - de^{-\beta x}\cos\beta x\right] \qquad \text{...eqn. 17.21}$$

The same result may be obtained from the first principle also.

QUESTIONS

1. Derive the equations of equilibrium for a symmetrically loaded cylindrical pipes.
2. The equation of equilibrium for a symmetrically loaded pipe is

$$\frac{\partial^4 w}{\partial x^4} + 4\beta^4 w = \frac{Z}{D}$$

where $$4\beta^4 = \frac{Eh}{R^2 D}.$$

Determine the complementary function for it.

3. Show that the particular solution in the equilibrium equation of symmetrically loaded pipe may be replaced by membrane solution.
4. A long cylindrical pipe is subjected to edge moment M_0 and edge shear Q_0 per unit circumferential length. Determine the expression for displacement.

Chapter

18

Membrane Theory for Shells of Revolutions

Middle surface of a shell of revolution is obtained by rotating a plane curve around an axis, which is denoted as shell axis. In this chapter, analysis of such shells by membrane theory is presented.

18.1 GEOMETRY OF SHELL OF REVOLUTION

Figure 18.1 shows a typical shell of revolution. By bisecting the middle surface of the shell with two series of plane, one containing the axis and another perpendicular to the axis, we create a net of lines. These network of curves may be called as meridians and parallels.

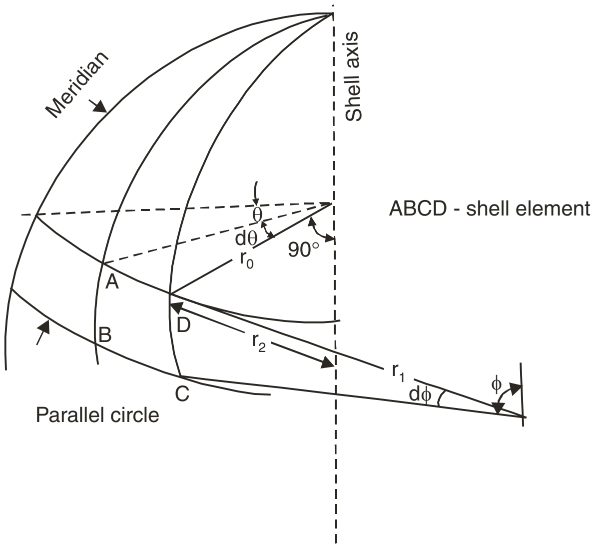


Fig. 18.1

Let,

r_1 — radius of curvature of the meridian.
r_2 — length of the shell normal to meridian at point A upto the shell axis.
r_0 — radius of curvature of parallel circle.

ϕ — the angle between the shell axis and shell normal.

θ — the angle in the plane of parallel circle between a reference direction and the radius joining.

It may be noted that the angle between the shell axis and the radius of the parallel circle r_0 is 90°.

$$\therefore \qquad r_0 = r_2 \sin \phi \qquad \text{...eqn. 18.1}$$

18.2 EQUATIONS OF EQUILIBRIUM

Consider an element formed by two meridians and two parallels as shown in Fig. 18.2.

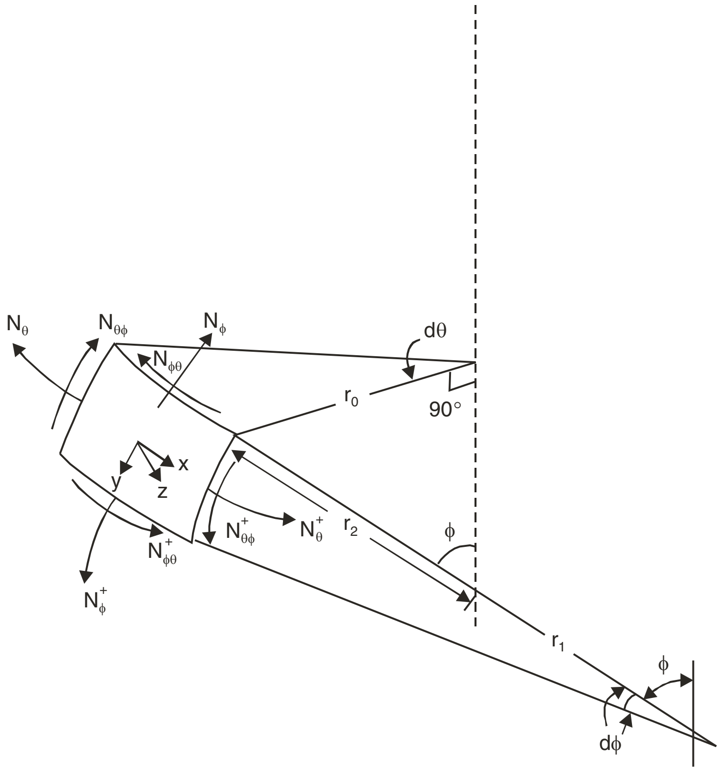


Fig. 18.2

Let N_θ, N_ϕ, $N_{\theta\phi}$ and $N_{\phi\theta}$ be the forces per unit length. Then obviously,

$$N_\theta^+ = N_\theta + \frac{\partial N_\theta}{\partial \theta} d\theta$$

$$N_\phi^+ = N_\phi + \frac{\partial N_\phi}{\partial \phi} d\phi \qquad \text{...eqn. 18.2}$$

$$N_{\theta\phi}^+ = N_{\theta\phi} + \frac{\partial N_{\theta\phi}}{\partial \theta} d\theta \text{ and}$$

$$N_{\phi\theta}^+ = N_{\phi\theta} + \frac{\partial N_{\phi\theta}}{\partial \phi} d\phi$$

Let,

X — Component of load in θ-direction per unit area.

Y — Component of load in ϕ-direction per unit area.

Z — Component of load in direction normal to the element in the inward direction, per unit area.

Noting that in membrane theory bending moment and transverse shear are neglected, the element is in equilibrium under the action of N_θ, N_ϕ, $N_{\theta\phi}$ and $N_{\phi\theta}$.

Three equations of equilibrium may be derived for the element by considering the forces in three mutually perpendicular directions θ, ϕ and Z.

1. Equilibrium of forces in θ direction

Component of various forces in θ-direction are:

(*a*) N_θ forces:

$$-N_\theta r_1 d\phi + \left(N_\theta + \frac{\partial N_\theta}{\partial \theta} d\theta\right) r_1 d\phi$$

$$= r_1 \frac{\partial N_\theta}{\partial \theta} \cdot d\theta .$$

(*b*) N_ϕ forces – No component.

(*c*) $N_{\phi\theta}$ forces:

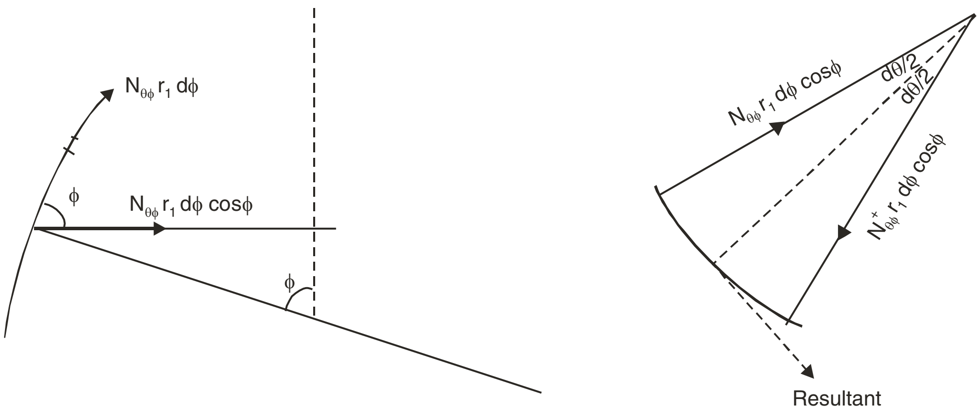
$$-N_{\phi\theta} r_0 d\theta + \left(N_{\phi\theta} + \frac{\partial N_{\phi\theta}}{\partial \phi} d\phi\right)\left(r_0 + \frac{\partial r_0}{\partial \phi} d\phi\right) d\theta$$

$$= N_{\phi\theta} \frac{\partial r_0}{\partial \phi} d\phi d\theta + r_0 \frac{\partial N_{\phi\theta}}{\partial \phi} d\phi d\theta$$

$$= \frac{\partial}{\partial \phi}\left(r_0 N_{\phi\theta}\right) d\phi\, d\theta$$

[Small quantity of higher order neglected].

(*d*) $N_{\theta\phi}$ forces:



(*a*) Vertical section (*b*) Plan view

Fig. 18.3

Referring to Fig. 18.3,

On face *AB* of the element, upward force

$$= N_{\theta\phi}\, r_1\, d\phi \qquad \text{[Ref. Fig. 18.3(a)]}$$

Its radial inward component

$$= N_{\theta\phi}\, r_1\, d\phi \cos\phi \qquad \text{[Ref. Fig. 18.3(b)]}$$

Similarly on face *CD,* downward force

$$= \left(N_{\theta\phi} + \frac{\partial N_{\theta\phi}}{\partial \theta} d\theta\right) r_1 d\phi$$

Its radial inward component,

$$= \left(N_{\theta\phi} + \frac{\partial N_{\theta\phi}}{\partial \theta} d\theta\right) r_1 d\phi \cos\phi \qquad \text{[Ref. Fig. 18.3(b)]}$$

Hence, the component of these $N_{\theta\phi}$ forces in 'θ' direction

$$= \left(N_{\theta\phi} + \frac{\partial N_{\theta\phi}}{\partial \theta} d\theta\right) r_1 d\phi \cos\phi \cdot \frac{d\theta}{2} + N_{\theta\phi} r_1 d\phi \cos\phi \cdot \frac{d\theta}{2}$$

$$= N_{\theta\phi} r_1 \cos\phi\, d\phi\, d\theta$$

[After neglecting small quantities of higher order].

(*e*) Load Component.

$$= X\, r_1 d\phi\, r_0\, d\theta$$

$$= X\, r_0 r_1\, d\theta d\phi$$

∴ The equation of equilibrium is,

$$r_1 \frac{\partial N_\theta}{\partial \theta} d\theta d\phi + \frac{\partial}{\partial \phi}\left(r_0 N_{\theta\phi}\right) d\theta d\phi + N_{\theta\phi} r_1 \cos\phi d\theta d\phi + X r_0 r_1 d\theta d\phi = 0$$

i.e.

$$\frac{\partial N_\theta}{\partial \theta} r_1 + N_{\theta\phi} r_1 \cos\phi + \frac{\partial}{\partial \phi}\left(N_{\theta\phi} r_0\right) + X r_1 r_0 = 0 \qquad \text{...eqn. 18.3}$$

2. Equation of Equilibrium of forces in ϕ direction.

(*a*) Component of N_ϕ forces in ϕ direction:

$$-N_\phi r_0 d\theta + \left(N_\phi + \frac{\partial N_\phi}{\partial \phi} d\phi\right)\left(r_0 + \frac{\partial r_0}{\partial \phi} d\phi\right) d\theta$$

$$= N_\phi \frac{\partial r_0}{\partial \phi} d\phi\, d\theta + r_0 N_\phi d\phi d\theta$$

$$= \frac{\partial}{\partial \phi}\left(r_0 N_\phi\right) d\phi\, d\theta$$

[Small quantity of higher order neglected].

(*b*) Component of N_θ forces:

Looking at Fig. 18.4(*a*), tangential outward forces in parallel plane are $N_\theta r_1 d\phi$ and $N_\theta^+ r_1\, d\phi$.

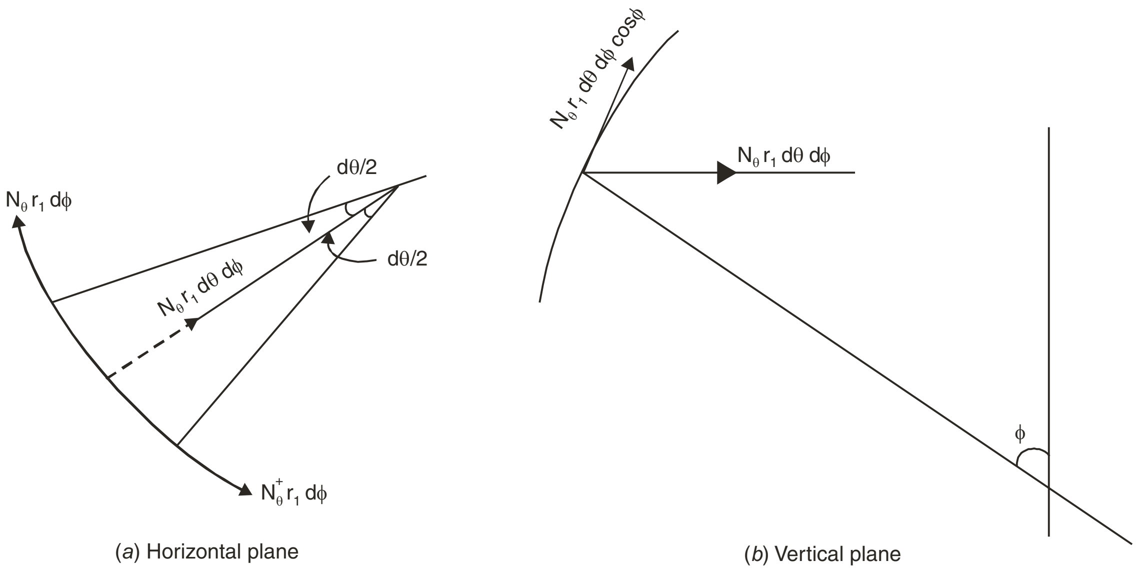


Fig. 18.4

Their radial inward component

$$= N_\theta r_1 d\phi \frac{d\theta}{2} + N_\theta^+ r_1 d\phi \frac{d\theta}{2}$$

$$= N_\theta r_1 d\phi\, d\theta$$

[After neglecting small quantity of higher order]

∴ Its component in ϕ-direction = $-\, N_\theta r_1 \cos\phi\; d\phi\; d\theta$ (Ref. Fig. 18.4(*b*))

(*c*) $N_{\phi\theta}$ forces – no component.

(*d*) Component of $N_{\theta\phi}$ forces:

$$= -N_{\theta\phi} r_1 d\phi + \left(N_{\theta\phi} + \frac{\partial N_{\theta\phi}}{\partial\theta} d\theta \right) r_1 d\phi$$

$$= \frac{\partial N_{\theta\phi}}{\partial\theta} \cdot r_1 d\theta\, d\phi$$

(*e*) Component of load:

$$= Y\, r_0 r_1\, d\theta\, d\phi$$

∴ Equation of equilibrium is

$$\frac{\partial}{\partial\phi}\left(N_\phi r_0\right) - N_\theta r_1 \cos\phi + \frac{\partial N_{\theta\phi}}{\partial\theta} r_1 + Y r_0 r_1 = 0 \qquad \text{...eqn. 18.4}$$

3. Equation of Equilibrium of forces in *z*-direction

(*a*) Component of N_θ forces:

From Fig. 18.5(*a*), it is clear that, there is a component $N_\theta r_1 d\phi\ d\theta$ in the horizontal plane. From Fig. 18.5(*b*), it is clear that, the component in *z*-direction

$$= N_\theta r_1 d\phi\ d\theta\ \sin\phi.$$

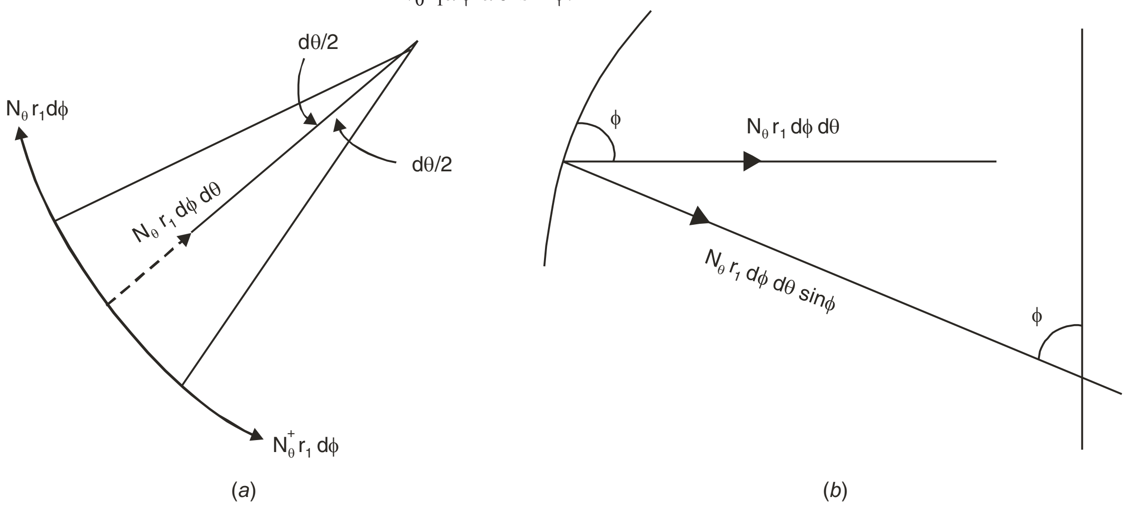


Fig. 18.5

(*b*) Component of N_ϕ forces in *z*-direction.

Referring to Fig. 18.6, we find component of N_ϕ forces in *z*-direction

$$= N_\phi r_0 d\theta \cdot \frac{d\phi}{2} + \left(N_\phi + \frac{\partial N_\phi}{\partial \phi} d\phi\right)\left(r_0 + \frac{\partial r_0}{\partial \phi} d\phi\right)\frac{d\theta\, d\phi}{2}$$

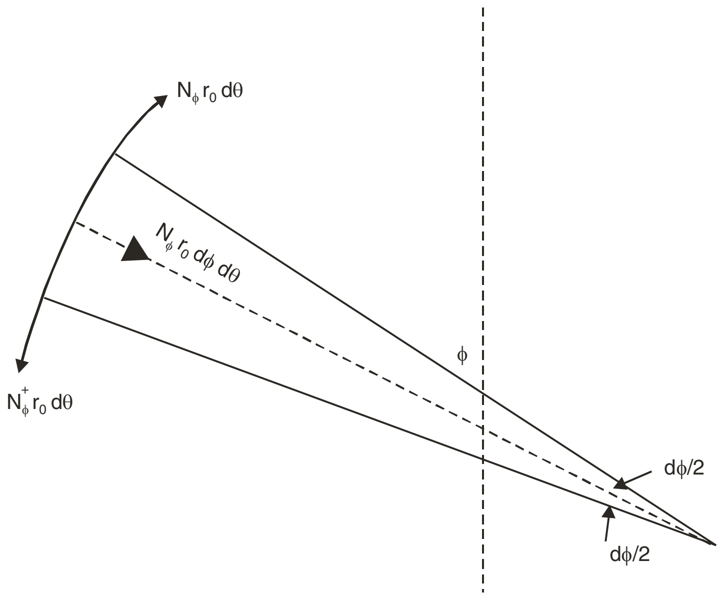


Fig. 18.6

$= N_\phi r_0 d\phi\, d\theta$

[Small quantities of higher order neglected]

(*c*) $N_{\theta\phi}$ and $N_{\phi\theta}$ have no components.

(*d*) Component of external load:

$= Z\, r_0 r_1 d\theta d\phi$

∴ The equation of equilibrium is

$$N_\theta r_1 \sin\phi + N_\phi r_0 + Z\, r_0 r_1 = 0$$

or

$$\frac{N_\theta \sin\phi}{r_0} + \frac{N_\phi}{r_1} + Z = 0$$

Since, $r_0 = r_2 \sin\phi$, we get,

$$\frac{N_\phi}{r_1} + \frac{N_\theta}{r_2} + Z = 0 \qquad \text{...eqn. 18.5}$$

Thus, the equations of equilibrium are:

$$\mathbf{r_1 \frac{\partial N_\theta}{\partial \theta} + \frac{\partial}{\partial \phi}\left(r_0 N_{\phi\theta}\right) + r_1 N_{\theta\phi} \cos\phi + r_0 r_1 X = 0}$$

$$\mathbf{\frac{\partial}{\partial \phi}\left(r_0 N_\phi\right) - r_1 N_\theta \cos\phi + r_1 \frac{\partial N_{\theta\phi}}{\partial \theta} + r_0 r_1 Y = 0} \qquad \text{...eqn. 18.6}$$

$$\mathbf{\frac{N_\phi}{r_1} + \frac{N_\theta}{r_2} + Z = 0.}$$

18.3 EQUATIONS OF EQUILIBRIUM FOR AXI-SYMMETRICALLY LOADED SHELLS

When in addition to the symmetry of forms, shells of revolution are subjected to axi-symmetrical loads, then,

(*i*) The forces are independent of θ *i.e.* all terms involving differentiation w.r.t. θ will vanish. Thus,

$N_\theta^+ = N_\theta$

$N_{\theta\phi}^+ = N_{\theta\phi}$

(*ii*) Shearing forces cannot exist, because if they exist there will be variation of forces in φ direction as well as in θ-direction. But due to symmetry N_θ cannot vary. Hence,

$N_{\theta\phi} = N_{\phi\theta} = 0.$

(*iii*) Due to symmetry load component $X = 0$. Hence, the forces on the element are as shown in Fig. 18.7.

There are only two forces N_ϕ and N_θ. They can be found by writing equations of equilibrium in φ-direction and *Z*-direction.

1. **Σ Forces in φ-direction = 0:**

(*a*) Component of N_ϕ forces:

$$-N_\phi r_0 d\theta + \left(N_\phi + \frac{\partial N_\phi}{\partial \phi} d\phi\right) d\theta \left(r_0 + \frac{\partial r_0}{\partial \phi} d\phi\right)$$

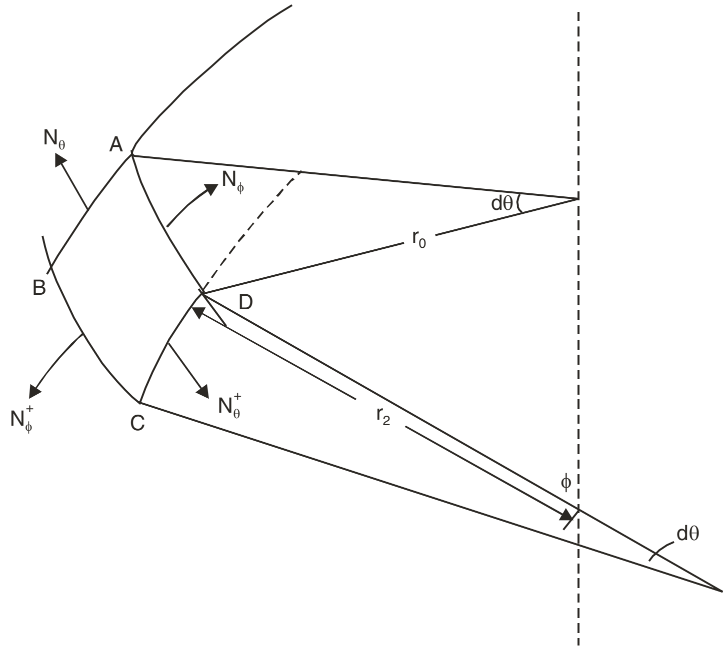


Fig. 18.7

$$= \frac{\partial N_\phi}{\partial \phi} r_0 d\phi d\theta + N_\phi \frac{\partial r_0}{\partial \phi} d\phi\, d\theta$$

$$= \frac{\partial}{\partial \phi}\left(r_0 N_\phi\right) d\phi\, d\theta$$

(*b*) Component of N_θ forces: Referring to Fig. 18.4, component of N_θ forces

$$= -N_\theta r_1 \cos\phi\, d\phi\, d\theta \cdot$$

(*c*) Load component

$$= Y r_0 r_1 d\theta\, d\phi \cdot$$

∴ The equation of equilibrium is,

$$\frac{\partial}{\partial \phi}\left(r_0 N_\phi\right) - r_1 N_\theta \cos\phi + r_0 r_1 Y = 0 \qquad \text{...eqn. 18.7}$$

2. $\sum$ Forces in Z-direction = 0:
As derived in Art 18.2.3, this equation is

$$\frac{N_\phi}{r_1} + \frac{N_\theta}{r_2} + Z = 0 \qquad \text{...eqn. 18.8}$$

18.4 SOLUTION OF EQUATIONS OF EQUILIBRIUM

By solving equations 18.7 and 18.8, the membrane forces N_θ and N_ϕ can be found.

From eqn. 18.8,

$$N_\theta = -Zr_2 - N_\phi \frac{r_2}{r_1}$$

Substituting it is eqn. 18.7, we get,

$$\frac{\partial}{\partial\phi}\left(r_0 N_\phi\right) + Zr_1r_2\cos\phi + N_\phi r_2\cos\phi + r_0r_1Y = 0$$

Multiplying each term by sinϕ, we get,

$$\frac{\partial}{\partial\phi}\left(r_0 N_\phi\right)\times\sin\phi + Zr_1r_2\cos\phi\sin\phi + N_\phi r_2\cos\phi\sin\phi + r_0r_1Y\sin\phi = 0$$

Noting that $r_0 = r_2$ sinϕ, we get,

$$\frac{\partial}{\partial\phi}\left(r_0 N_\phi\right)\times\sin\phi + N_\phi r_0\cos\phi = -r_1r_2\left(Z\cos\phi + Y\sin\phi\right)\sin\phi$$

i.e.
$$\frac{\partial}{\partial\phi}\left(r_0 N_\phi\sin\phi\right) = -r_1r_2\left(Z\cos\phi + Y\sin\phi\right)\sin\phi$$

$$r_0 N_\phi\sin\phi = -\int r_1r_2\left(Z\cos\phi + Y\sin\phi\right)\sin\phi d\phi + C$$

i.e.
$$r_2 N_\phi\sin^2\phi = -\int r_1r_2\left(Z\cos\phi + Y\sin\phi\right)\sin\phi d\phi + C$$

i.e.
$$N_\phi = -\frac{1}{r_2\sin^2\phi}\left[\int r_1r_2\left(Z\cos\phi + Y\sin\phi\right)\sin\phi d\phi + C\right]$$

To get physical meaning for it, let us multiply numerator and denominator by 2π.

$$N_\phi = -\frac{1}{2\pi r_2\sin^2\phi}\left[\int 2\pi r_1r_2\left(Y\sin\phi + Z\cos\phi\right)\sin\phi d\phi + C\right] \qquad \text{...eqn. 18.9}$$

where C is new arbitrary constant.

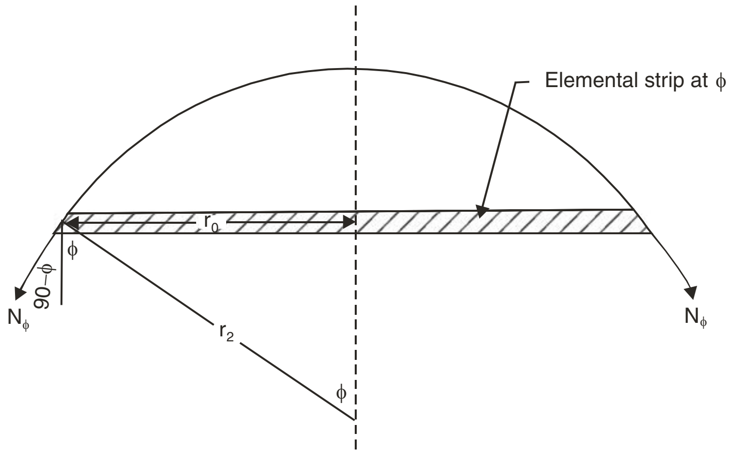


Fig. 18.8

A simple physical meaning of the above equation is possible. Referring to Fig. 18.8, the term $2\pi r_1 r_2 \sin\phi\, d\phi$ stands for the surface area of an elemental strip of the shell. The term $(Y \sin\phi + Z \cos\phi)$ stands for the vertical component of the forces per unit area acting on this elemental strip. Hence, the term $2\pi r_1 r_2 \sin\phi(Y \sin\phi + Z \cos\phi)$ stands for the vertical load acting on the strip. The integral $\int 2\pi r_1 r_2 \sin\phi\, (Y \sin\phi + Z \cos\phi) d\phi$ represents the vertical load on the shell upto the level where the meridian angle is ϕ.

$2\pi r_2 \sin^2\phi\, N_\phi$ represents the vertical component of N_ϕ acting around the circle of latitude ϕ. Thus, the equation 18.9 is nothing but a mathematical statement of the vertical equilibrium of portion of shell above the parallel circle at ϕ. Hence, it can be stated as

$$N_\phi = -\frac{W}{2\pi r_2 \sin^2\phi} \qquad \text{...eqn. 18.10}$$

where W is the total vertical load acting on the dome above the level denoted by ϕ.

The constant of integration C can be made use of to account for concentrated load, if any, applied at the crown or as a ring load above this level. If no such concentrated load exists, $C = 0$.

Example 18.1. Determine the membrane forces in a hemispherical shell subjected to self weight only.

Solution. Let the self weight per unit area be 'g' and radius of shell be 'a'. [Ref. Fig. 18.9(a)]

Area of shell above ϕ

$$= \int_0^\phi a\, d\beta\, 2\pi r_0$$

$$= \int_0^\phi a \cdot 2\pi \cdot a \sin\beta \cdot d\beta$$

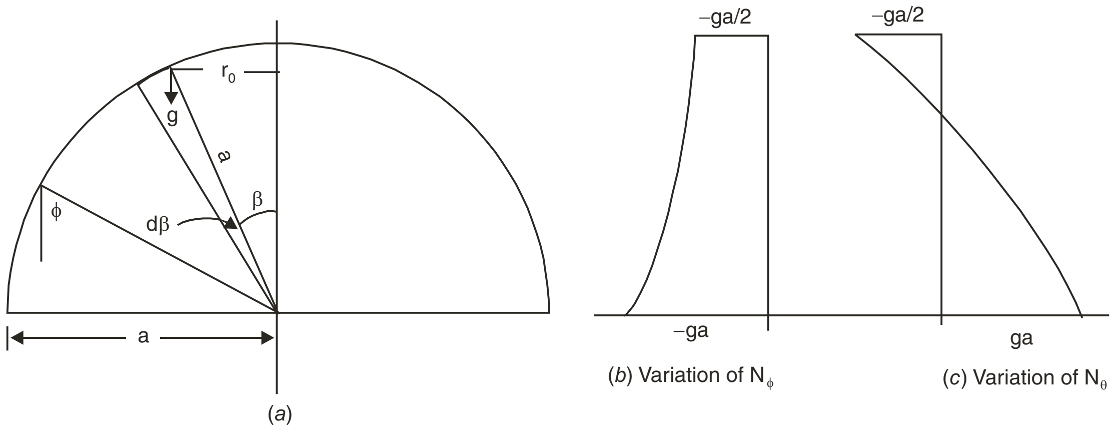


Fig. 18.9

$$= 2\pi a^2 [-\cos\beta]_0^\phi$$

$$= 2\pi a^2 (1 - \cos\phi)$$

Total vertical downward load above the level of ϕ is

$$w = 2\pi a^2 g[1 - \cos\phi]$$

$$N_\phi = -\frac{2\pi a^2 g(1-\cos\phi)}{2\pi r_0 \sin\phi}$$

$$= -\frac{2\pi a^2 g(1-\cos\phi)}{2\pi a \sin^2\phi}, \qquad \text{since } r_0 = a\sin\phi$$

$$= -\frac{ga(1-\cos\phi)}{1-\cos^2\phi}$$

$$= -\frac{ga}{1+\cos\phi} \qquad \text{...(Ans.)}$$

To find N_θ,

$$Z = g\cos\phi$$

$\therefore$ From the equation of equilibrium,

$$\frac{N_\phi}{r_1} + \frac{N_\theta}{r_2} + g\cos\phi = 0, \text{ we get,}$$

$$-\frac{ga}{a(1+\cos\phi)} + \frac{N_\theta}{a} + g\cos\phi = 0 \qquad [\text{Since } r_1 = r_2 = a \text{ for a spherical shell}]$$

$\therefore$
$$N_\theta = a\cdot g\cdot\left(\frac{1}{1+\cos\phi} - \cos\phi\right) \qquad \text{...(Ans.)}$$

The variations of N_ϕ and N_θ forces for a hemispherical shell are as shown in Fig. 18.9(*b*) and 18.9(*c*).

Example 18.2. Determine membrane forces in a hemispherical shell due to concentrated load at crown only.

Solution. Let P be the concentrated load at crown and 'a' be the radius of hemispherical shell.

$\therefore$ The vertical equilibrium of the shell above the parallel circle at ϕ gives,

$$N_\phi = -\frac{P}{2\pi a \sin^2\phi} \qquad \text{...(Ans.)}$$

Then, to find N_θ, we know $Z = 0$ for this loading.

$\therefore$
$$\frac{N_\phi}{a} + \frac{N_\theta}{a} + 0 = 0.$$

i.e.
$$N_\theta = -N_\phi = \frac{P}{2\pi a \sin^2\phi} \qquad \text{...(Ans.)}$$

Example 18.3. Determine the membrane forces in a hemispherical shell due to snow load only.

Solution. Let snow load be p_0 per unit horizontal area of shell surface. Hence, the total load upto the circle of latitude ϕ is,

$$W = p_0 \pi r_0^2$$
$$= p_0 \pi a^2 \sin^2\phi$$

$$\therefore \quad N_\phi = -\frac{W}{2\pi r_0 \sin\phi} = -\frac{W}{2\pi a \sin^2\phi}$$

$$= -\frac{p_0 \pi a^2 \sin^2\phi}{2\pi a \sin^2\phi}$$

$$= -\frac{p_a a}{2} \qquad \text{...(Ans.)}$$

Now, intensity of load on surface at ϕ

$$= \frac{p_0 \times 1}{\dfrac{1}{\cos\phi}} = p_0 \cos\phi$$

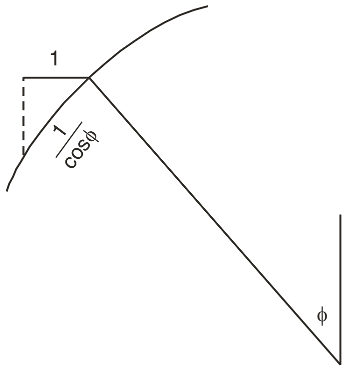


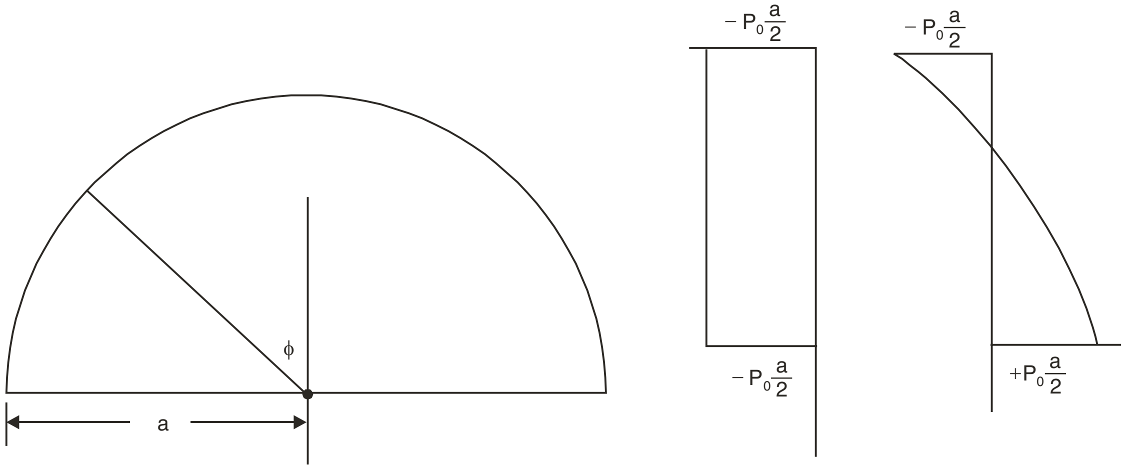
Fig. 18.10

$$\therefore \quad Z = p_0 \cos\phi \cdot \cos\phi$$
$$= p_0 \cos^2\phi$$

$\therefore$ From the equation of equilibrium,

$$\frac{N_\phi}{r_1} + \frac{N_\theta}{r_2} + Z = 0, \text{ we get,}$$

$$-\frac{p_0 a}{2} + \frac{N_\theta}{a} + p_0 \cos^2\phi = 0$$



(*a*) (*b*) Variation of N_ϕ (*c*) Variation of N_θ

Fig. 18.11

$$N_\theta = -\left(\frac{p_0 a}{2 \times a} - p_0 \cos^2 \phi\right) \times a$$

$$= \frac{ap_0}{2}\left(1 - 2\cos^2 \phi\right)$$

$$= +\frac{p_0 a}{2} \cos 2\phi \qquad \text{...(Ans.)}$$

The variation of membrane forces are shown in Fig. 18.11.

18.5 MEMBRANE ANALYSIS OF CONICAL SHELLS

Referring to Fig. 18.12, it is clear that ϕ cannot be taken as a coordinate. Let the base angle be α. Taking 's' which is the distance from the apex to the point under consideration, as a coordinate, we note,

$$\phi = \alpha$$
$$r_0 = s \cos \alpha$$

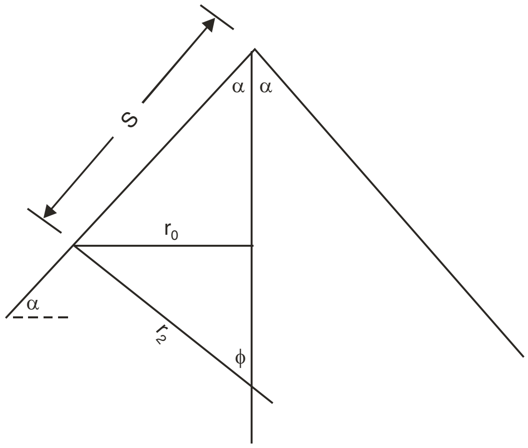


Fig. 18.12 Conical Shell

$$r_1 = \alpha, \text{ and}$$
$$r_2 = s \cot \alpha$$

$\therefore$
$$N_S = -\frac{W}{2\pi r_0 \sin \phi}$$

$$= -\frac{W}{2\pi s \cos \alpha \sin \alpha} \qquad \text{...eqn. 18.11}$$

and
$$\frac{N_S}{\alpha} + \frac{N_\theta}{s \cot \alpha} + Z = 0.$$

$\therefore$
$$N_\theta = -Zs \cot \alpha \qquad \text{...eqn. 18.12}$$

Example 18.4. Find the membrane forces in the umbrella type conical shell shown in Fig. 18.13.

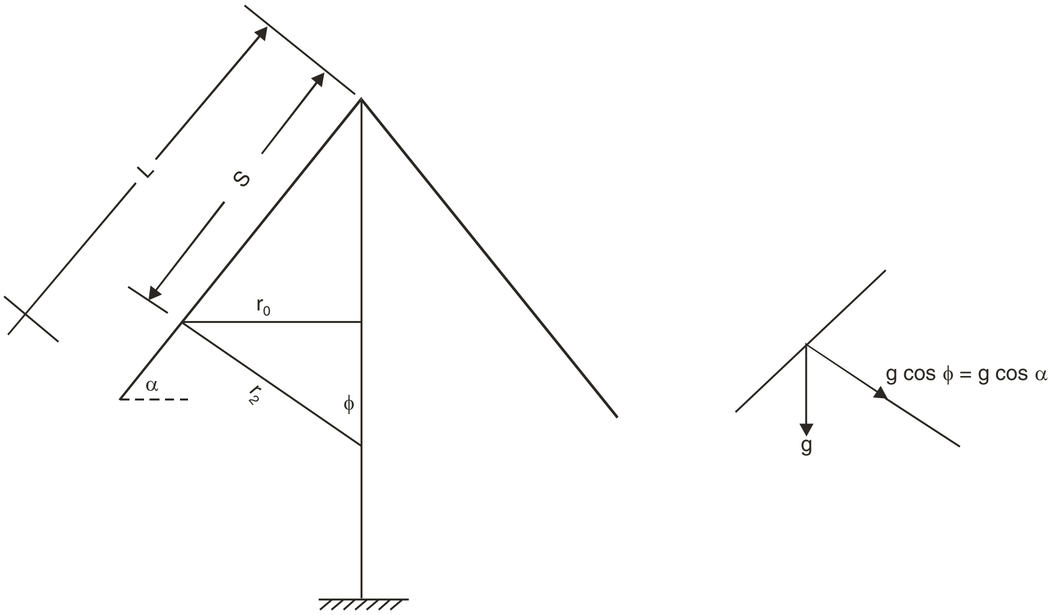


Fig. 18.13

Solution. Load below a horizontal section at a distance '*s*' from apex

$$W = \int_S^L 2\pi s \cdot \cos\alpha \, ds \, g$$

$$= 2\pi g \left[\frac{s^2}{2}\right]_S^L \cos\alpha$$

$$= \pi g \left(L^2 - S^2\right) \cos\alpha$$

∴ $$N_S = + \frac{W}{2\pi s \cos\alpha \sin\alpha}$$

$$= + \frac{\pi g \left(L^2 - S^2\right) \cos\alpha}{2\pi S \cos\alpha \sin\alpha}$$

$$= + \frac{g\left(L^2 - S^2\right)}{2S \sin\alpha}$$ ...**(Ans.)**

Now $Z = g \cos\alpha$. (Ref. Fig. 18.13)

∴ $N_\theta = -Zs \cot\alpha$

$= -g \cos\alpha \cdot s \cdot \cot\alpha$

$= -gs \cos\alpha \cdot \cot\alpha$ ...**(Ans.)**

18.6 ROTATIONAL HYPERBOLOID OF ONE SHEET

This type of shells are commonly used as cooling towers in thermal stations. If a hyperbolic curve is rotated about a vertical axis, passing through the pole, such shells are generated.

Figure 18.14 shows a typical rotational hyperboloid of one sheet.

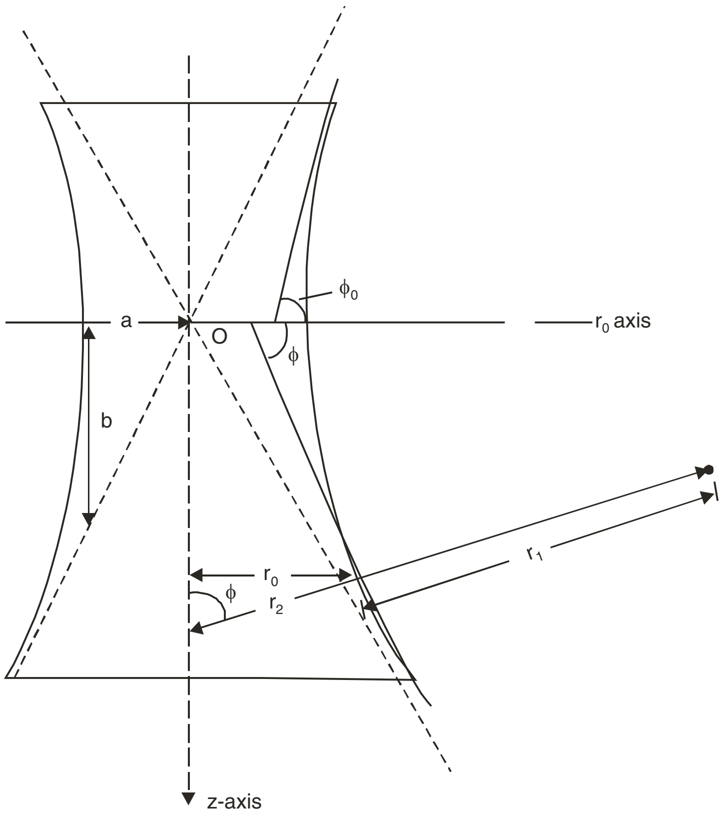


Fig. 18.14

with r_0 and z-axis selected as shown in Fig. 18.14, the equation of hyperbola is

$$\frac{r_0^2}{a^2} - \frac{z^2}{b^2} = 1$$

i.e.

$$r_0 = a\sqrt{1 + \frac{z^2}{b^2}} \quad \text{...eqn. 18.13}$$

The principal radii of curvatures r_1 and r_2 are given by,

$$r_1 = \frac{\left[1 + \left(f'(z)\right)^2\right]^{3/2}}{f'(z)} \quad \text{...eqn. 18.14}$$

and

$$r_2 = f(z)[1 + (f'(z))^2]^{1/2} \quad \text{...eqn. 18.15}$$

where

$$f(z) = r_0$$

Now, $$f(z) = r_0 = a\sqrt{1+\frac{z^2}{b^2}}$$

$\therefore$ $$f'(z) = \frac{\partial r_0}{\partial z} = -a\frac{1}{2}\frac{2\frac{z}{b^2}}{\sqrt{1+\frac{z^2}{b^2}}} = -\frac{a}{b^2}\frac{z}{\sqrt{1+\frac{z^2}{b^2}}}$$

$$= -\frac{a^2}{b^2}\frac{z}{r_0}$$

$$f''(z) = -\frac{a}{b^2}\left[\frac{1}{\sqrt{1+\frac{z^2}{b^2}}} + z\frac{\frac{-1}{2}\frac{2z}{b^2}}{\left(1+\frac{z^2}{b^2}\right)^{3/2}}\right]$$

$$= -\frac{a}{b^2}\left[\frac{1+\frac{z^2}{b^2}-\frac{z^2}{b^2}}{\left(1+\frac{z^2}{b^2}\right)^{3/2}}\right]$$

$$= -\frac{a}{b^2}\frac{1}{\left(\frac{r_0^2}{a^2}\right)^{3/2}}$$

$$= -\frac{a^4}{b^2 r_0^3}$$

$\therefore$ $$r_1 = \frac{\left[1+(f_2')^2\right]^{3/2}}{f''(z)}$$

$$= \frac{\left[1+\frac{a^4 z^4}{b^4 r_0^3}\right]^{3/2}}{-a^4/(b^2 r_0^3)}$$

$$= -\frac{b^2}{a^4} r_0^3 \left[1+\frac{a^4 z^4}{b^4 r_0^3}\right]^{3/2}$$

$$= -\frac{b^2}{a^4} \times a^6 \left[\frac{r_0^2}{a^4} + \frac{z^4}{b^4} \right]^{3/2}$$

$$= -a^2 b^2 \left[\frac{r_0^2}{a^4} + \frac{z^4}{b^4} \right]^{3/2} \qquad \text{...eqn. 18.16}$$

$$r_z = f(z)\left[1 + \left(f'(z)\right)^2\right]^{1/2}$$

$$= r_0 \left[1 + \frac{a^4 z^2}{b^4 r_0^2}\right]^{1/2}$$

$$= a^2 \left[\frac{r_0^2}{a^4} + \frac{z^2}{b^4} \right]^{1/2} \qquad \text{...eqn. 18.17}$$

From eqns. 18.16 and 18.17, we get,

$$r_1 = -\frac{b^2}{a^4} r_2^3 \qquad \text{...eqn. 18.18}$$

The equation of hyperbola (Eqn. 18.13) may be rewritten as

$$z = \pm \frac{b}{a} \sqrt{r_0^2 - a^2}$$

$$\therefore \qquad \frac{\partial z}{\partial r_0} = \tan\phi = \frac{b}{a}\frac{1}{2}\left(r_0^2 - a^2\right)^{-1/2} \times 2r_0$$

$$= \frac{b}{a} \frac{r_0}{\left(r^2 - a^2\right)^{1/2}}$$

$$\therefore \qquad \cot\phi = \pm \frac{a}{b} \frac{\left(r_0^2 - a^2\right)^{1/2}}{r_0}$$

$$\therefore \qquad \cot^2\phi = \frac{a^2}{b^2} \frac{\left(r_0^2 - a^2\right)}{r_0^2} = \frac{a^2}{b^2}\left(1 - \frac{a^2}{r_0^2}\right)$$

$$\frac{b^2}{a^2}\cot^2\phi = 1 - \frac{a^2}{r_0^2}$$

$$\therefore \qquad \frac{a^2}{r_0^2} = 1 - \frac{b^2}{a^2}\cot^2\phi = \frac{a^2 \sin^2\phi - b^2 \cos^2\phi}{a^2 \sin^2\phi}$$

$$r_0^2 = \frac{a^4 \sin^2 \phi}{a^2 \sin^2 \phi - b^2 \cos^2 \phi}$$

or $$r_0 = \frac{a^2 \sin \phi}{\left(a^2 \sin^2 \phi - b^2 \cos^2 \phi\right)^{1/2}} \qquad \text{...eqn. 18.19}$$

But $$r_0 = r_2 \sin \phi$$

Hence, $$r_2 = \frac{r_0}{\sin \phi} = \frac{a^2}{\left(a^2 \sin^2 \phi - b^2 \cos^2 \phi\right)^{1/2}} \qquad \text{...eqn. 18.20}$$

and $$r_1 = -\frac{b^2}{a^4} r_2^3 = -\frac{a^2 b^2}{\left(a^2 \sin^2 \phi - b^2 \cos^2 \phi\right)^{3/2}} \qquad \text{...eqn. 18.21}$$

After noting the above geometric relations of hyperbola, it is possible to analyse the rotational hyperboloid of one sheet for any type of loading.

Example 18.5. Analyse a typical rotational hyperboloid of one sheet subjected to self weight g/unit surface area and find the membrane forces.

Solution. Let W be the total load of the shell above level ϕ. Then,

$$W = g \int_{\phi}^{\phi_0} 2\pi r_0 r_1 d\phi$$

$$= 2\pi g \int_{\phi}^{\phi_0} \frac{a^2 \sin \phi}{\left(a^2 \sin^2 \phi - b^2 \cos^2 \phi\right)^{1/2}} \frac{\left(-b^2 a^2\right)}{\left(a^2 \sin^2 \phi - b^2 \cos^2 \phi\right)^{3/2}} d\phi$$

$$= -2\pi g a^4 b^2 \int_{\phi}^{\phi_0} \frac{\sin \phi}{\left(a^2 \sin^2 \phi - b^2 \cos^2 \phi\right)^2} d\phi$$

Let us substitute, $$\cos \phi = \frac{a}{\sqrt{a^2 + b^2}} \xi.$$

then $$-\sin \phi \, d\phi = \frac{a}{\sqrt{a^2 + b^2}} d\xi.$$

$\therefore$ $$W = 2\pi g a^4 b^2 \frac{a}{\sqrt{a^2 + b^2}} \int_{\phi}^{\phi_0} \frac{d\xi}{\left[a^2 \left(1 - \frac{a^2}{a^2 + b^2} \xi^2\right) - b^2 \frac{a^2}{a^2 + b^2} \xi^2\right]^2}$$

$$= 2\pi g \frac{a^5 b^2}{\sqrt{a^2+b^2}} \times \frac{1}{a^4} \int_{\phi}^{\phi_0} \frac{d\xi}{\left(1 - \frac{a^2+b^2}{a^2+b^2}\xi^2\right)^2}$$

$$= \frac{2\pi g\ ab^2}{\sqrt{a^2+b^2}} \int_{\phi}^{\phi_0} \frac{d\xi}{\left(1-\xi^2\right)^2}$$

$$= \frac{2\pi g\, ab^2}{\sqrt{a^2+b^2}} \times \frac{1}{4}\left[\frac{2\xi}{1-\xi^2} + \log\frac{1+\xi}{1-\xi}\right]_{\xi_0}^{\xi}$$

$$= \frac{\pi g\, ab^2}{2 \times \sqrt{a^2+b^2}}\left[f(\xi) - f(\xi_0)\right]$$

where $$f(\xi) = \frac{2\xi}{1-\xi^2} + \log\frac{1+\xi}{1-\xi}$$

∴ $$N_\phi = -\frac{W}{2\pi r_0 \sin\phi}$$

$$= -\frac{g}{4}\frac{ab^2}{\sqrt{a^2+b^2}}\frac{1}{r_0 \sin\phi}\left[f(\xi) - f(\xi_0)\right]$$

Now, $$r_0 \sin\phi = r_2 \sin^2\phi$$

$$= \frac{a^2 \sin^2\phi}{\left(a^2 \sin^2\phi - b^2\cos^2\phi\right)^{1/2}}$$

$$= \frac{a^2\left[1 - \frac{a^2}{a^2+b^2}\xi^2\right]}{\left[a^2\left(1 - \frac{a^2}{a^2+b^2}\xi^2\right) - b^2\frac{a^2}{a^2+b^2}\xi^2\right]^{1/2}}$$

$$= \frac{a^2\left[a^2+b^2-a^2\xi^2\right]/\left(a^2+b^2\right)}{a\left[1 - \frac{a^2+b^2}{a^2+b^2}\xi^2\right]^{1/2}}$$

$$= \frac{a\left(a^2+b^2-a^2\xi^2\right)}{\left(a^2+b^2\right)\left(1-\xi^2\right)^{1/2}}$$

$$\therefore \qquad N_\phi = -\frac{g}{4}\frac{ab^2}{\sqrt{a^2+b^2}}\frac{\left(a^2+b^2\right)\left(1-\xi^2\right)^{1/2}}{a\left(a^2+b^2-a^2\xi^2\right)}\left[f(\xi)-f(\xi_0)\right]$$

$$= -\frac{g}{4}b^2\sqrt{a^2+b^2}\,\frac{\sqrt{1-\xi^2}}{\left(a^2+b^2-a^2\xi^2\right)}\left[f(\xi)-f(\xi_0)\right] \qquad \text{...eqn. 18.22}$$

To find N_θ;

$$\frac{N_\phi}{r_1}+\frac{N_\theta}{r_2} = -g\cos\phi$$

$$N_\theta = -gr_2\cos\phi - \frac{r_2}{r_1}N_\phi$$

$$r_2 = \frac{a^2}{\left(a^2\sin^2\phi - b^2\cos^2\phi\right)^{1/2}}$$

$$= \frac{a^2}{\left(a^2\left(1-\cos^2\phi\right)-b^2\cos^2\phi\right)^{1/2}} = \frac{a^2}{\left[a^2-\left(a^2+b^2\right)\cos^2\phi\right]^{1/2}}$$

but $\left(a^2+b^2\right)\cos^2\phi = a^2\xi^2$

$$\therefore \qquad r_2 = \frac{a^2}{\left(a^2-a^2\xi^2\right)^{1/2}} = \frac{a}{\left(1-\xi^2\right)^{1/2}}$$

$$r_1 = -\frac{b^2}{a^4}r_2^3$$

$$= -\frac{b^2}{a^4}\times\frac{a^3}{\left(1-\xi^2\right)^{3/2}}$$

$$= \frac{-b^2}{a\left(1-\xi^2\right)^{3/2}}$$

$$\therefore \qquad N_\theta = -gr_2\cos\phi - \frac{r_2}{r_1}N_\phi$$

$$= -g\frac{a}{\left(1-\xi^2\right)^{1/2}}\cdot\frac{a}{\sqrt{a^2+b^2}}\xi + \frac{a}{\left(1-\xi^2\right)^{1/2}}\times\frac{a}{b^2}\left(1-\xi^2\right)^{3/2}N_\phi$$

$$= -\frac{ga^2}{\sqrt{a^2+b^2}}\frac{\xi}{\left(1-\xi^2\right)^{1/2}}+\frac{a^2}{b^2}\left(1-\xi^2\right)N_\phi \qquad \text{...eqn. 18.23}$$

Equations 18.22 and 18.23 give the membrane forces.

To prove, $\mathbf{I} = \int \frac{\mathbf{d\xi}}{\left(\mathbf{1}-\xi^2\right)^2} = \frac{\mathbf{1}}{\mathbf{4}}\left[\frac{\mathbf{2\xi}}{\mathbf{1}-\xi^2}+\log\frac{\mathbf{1}+\xi}{\mathbf{1}-\xi}\right]$:

Let, $\xi = \sin\theta$

$d\xi = \cos\theta \, d\theta$

$$I = \int \frac{\cos\theta \, d\theta}{\cos^4\theta} = \int \frac{1}{\cos^3\theta} d\theta = \int \sec^3\theta \, d\theta$$

$$= \int \sec^2\theta \cdot \sec\theta \cdot d\theta$$

$$= \int \frac{d}{d\theta}(\tan\theta) \cdot \sec\theta \cdot d\theta$$

$$= \sec\theta \tan\theta - \int \tan\theta \cdot \sec\theta \tan\theta \, d\theta$$

$$= \sec\theta \tan\theta - \int \sec\theta \, (\sec^2\theta - 1) \, d\theta$$

$$= \sec\theta \tan\theta - \int \sec^3\theta d\theta + \int \sec\theta \, d\theta$$

i.e. $2I = \sec\theta \tan\theta + \int \sec\theta \, d\theta$

$\therefore$ $$I = \frac{1}{2}\left[\sec\theta\tan\theta + \log\left(\sec\theta + \tan\theta\right)\right]$$

$$= \frac{1}{2}\left[\frac{\sin\theta}{\cos^2\theta} + \log\left(\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\xi}{1-\xi^2} + \log\frac{1+\xi}{\sqrt{1-\xi^2}}\right]$$

$$= \frac{1}{2}\left[\frac{\xi}{1-\xi^2} + \log\frac{1+\xi}{\sqrt{1-\xi} \times \sqrt{1+\xi}}\right]$$

$$= \frac{1}{2}\left[\frac{\xi}{1-\xi^2} + \log\frac{\sqrt{1+\xi}}{\sqrt{1-\xi}}\right]$$

$$= \frac{1}{2}\left[\frac{\xi}{1-\xi^2} + \frac{1}{2}\log\frac{1+\xi}{1-\xi}\right]$$

$$= \frac{1}{4}\left[\frac{2\xi}{1-\xi^2} + \log\frac{1+\xi}{1-\xi}\right].$$

QUESTIONS

1. Derive the equations of equilibrium for a shell of revolution, neglecting bending of the element.
2. A shell of revolution is subjected to symmetrical load. Considering membrane theory, determine the equations of equilibrium. Bring out physical meaning for the expression N_ϕ.
3. Determine the membrane forces in a hemispherical dome subject to
   (*i*) Self weight '*g*' per unit area
   (*ii*) Snow load '*p*' per unit projected area.
4. Determine membrane forces in an inverted umbrella type conical shell supported centrally and subjected to self weight only. Assume uniform thickness.

Chapter

19

Membrane Theory for Shells of Translation

Elliptic paraboloid, hyperbolic paraboloid and conoids are doubly curved shells generated by translation of one curve over the other. In this chapter, membrane theory of such shells is dealt.

19.1 ASSUMPTIONS

In developing the membrane theory, the following assumptions are made:

1. The thickness of the shell is small compared to other two dimension. In other words, the shell is treated as thin *i.e.* the stresses normal to shell surface are ignored.
2. The deformation is small compared to the thickness of the shell *i.e.* stresses in the middle surface are assumed zero.
3. Points on lines normal to the middle surface before deformation remain on the same normal even after deformation. In other words, shear deformations are ignored.
4. The material of the shell is homogeneous, isotropic and linearly elastic.
5. The thickness of the shell is uniform.

19.2 COORDINATE AXES

The curvilinear coordinates may be convenient to generate the geometry of the shell. But use of the consequent equations to satisfy the boundary conditions at the rectangular boundaries encountered in practice is very difficult. Hence, the cartesian coordinates are used. Right hand system of cartesian coordinates are used. Figure 19.1 shows a typical element in space and its projected element in x-y plane.

19.3 MONGE'S NOTATIONS

The surface is represented by $z = f(x, y)$. The following notations are used to write the equations conveniently. These notations are known as Monge's notations:

$$\frac{\partial z}{\partial x} = p, \frac{\partial z}{\partial y} = q$$

$$\frac{\partial^2 z}{\partial x^2} = r, \ \frac{\partial^2 z}{\partial x \partial y} = s, \ \frac{\partial^2 z}{\partial y^2} = t \qquad \text{...eqn. 19.1}$$

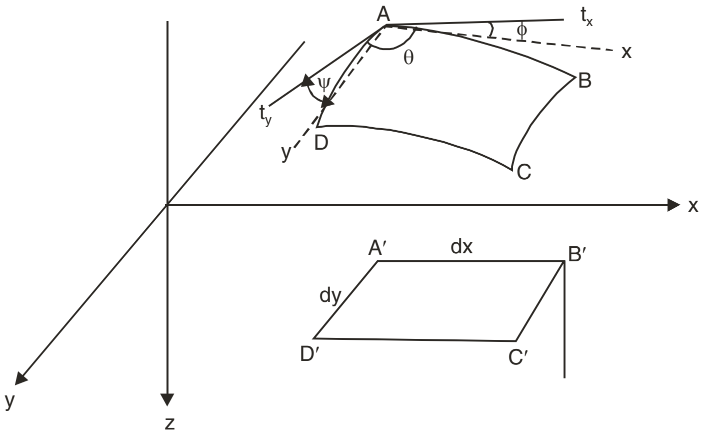


Fig. 19.1 Typical Element and Coordinate direction

19.4 PROPERTIES OF THE ELEMENT

Let at *A*,

t_x – Tangent to the surface in *x*-direction
t_y – Tangent to the surface in *y*-direction
ϕ – Angle between t_x and *x*-direction
ψ – Angle between t_y and *y*-direction
θ – Interior surface angle
ds_x – Surface length of element in *x*-direction
ds_y – Surface length of element in *y*-direction
dx – Projection of ds_x on *x*-*y* plane
dy – Projection of ds_y on *x*-*y* plane.

Referring to Figure 19.2,

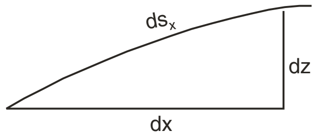


Fig. 19.2 Side *AB* of element in *x*-*z* plane

$$\therefore \quad ds_x^2 = dx^2 + dz^2$$

$$ds_x = \sqrt{dx^2 + dz^2}$$

$$= dx\sqrt{1 + \left(\frac{dz}{dx}\right)^2}$$

$$= dx\sqrt{1 + p^2} \qquad \text{...eqn. 19.2}$$

Similarly,

$$ds_y = \sqrt{dy^2 + dz^2}$$

$$= dy\sqrt{1+q^2} \quad \text{...eqn. 19.3}$$

$$\cos\phi = \frac{dx}{ds_x} = \frac{1}{\sqrt{1+p^2}} \quad \text{...eqn. 19.4}$$

$$\cos\psi = \frac{dy}{ds_y} = \frac{1}{\sqrt{1+q^2}} \quad \text{...eqn. 19.5}$$

$$\cos\theta = \sin\phi \cdot \sin\psi$$

$$= \sqrt{1-\cos^2\phi}\,\sqrt{1-\cos^2\psi}$$

$$= \sqrt{1-\frac{1}{1+p^2}}\,\sqrt{1-\frac{1}{1+q^2}}$$

$$= \sqrt{\frac{p^2}{1+p^2}}\,\sqrt{\frac{q^2}{1+q^2}}$$

$$= \frac{pq}{\sqrt{1+p^2}\,\sqrt{1+q^2}} \quad \text{...eqn. 19.6}$$

Surface area of the element

$$dA = ds_x\, ds_y \sin\theta$$

$$= dx\sqrt{1+p^2}\,dy\sqrt{1+q^2}\,\sqrt{1-\cos^2\theta}$$

$$= dx\,dy\sqrt{1+p^2}\,\sqrt{1+q^2}\,\sqrt{1-\frac{p^2q^2}{\left(1+p^2\right)\left(1+q^2\right)}}$$

$$= dx\,dy\sqrt{1+p^2}\,\sqrt{1+q^2}\,\frac{\sqrt{\left(1+p^2\right)\left(1+q^2\right)-p^2q^2}}{\left(\sqrt{1+p^2}\right)\left(\sqrt{1+q^2}\right)}$$

$$= dx\,dy\sqrt{1+p^2+q^2} \quad \text{...eqn. 19.7}$$

19.5 MEMBRANE ANALYSIS

Let the membrane forces in the element be N_x, N_y and N_{xy}.

In the analysis, it is convenient to use the components of membrane forces parallel to the axes *x* and *y,* imagined to act on the projected surface in *x-y* plane. These forces are called pseudo forces and they are denoted by '*n*' with suitable suffixes. Thus, the pseudo stress resultant n_x is such that it exert the same force in the x-direction on the projected side $A'D'$ as the membrane force does on side AD. Figure 19.3 shows the membrane forces and pseudo forces.

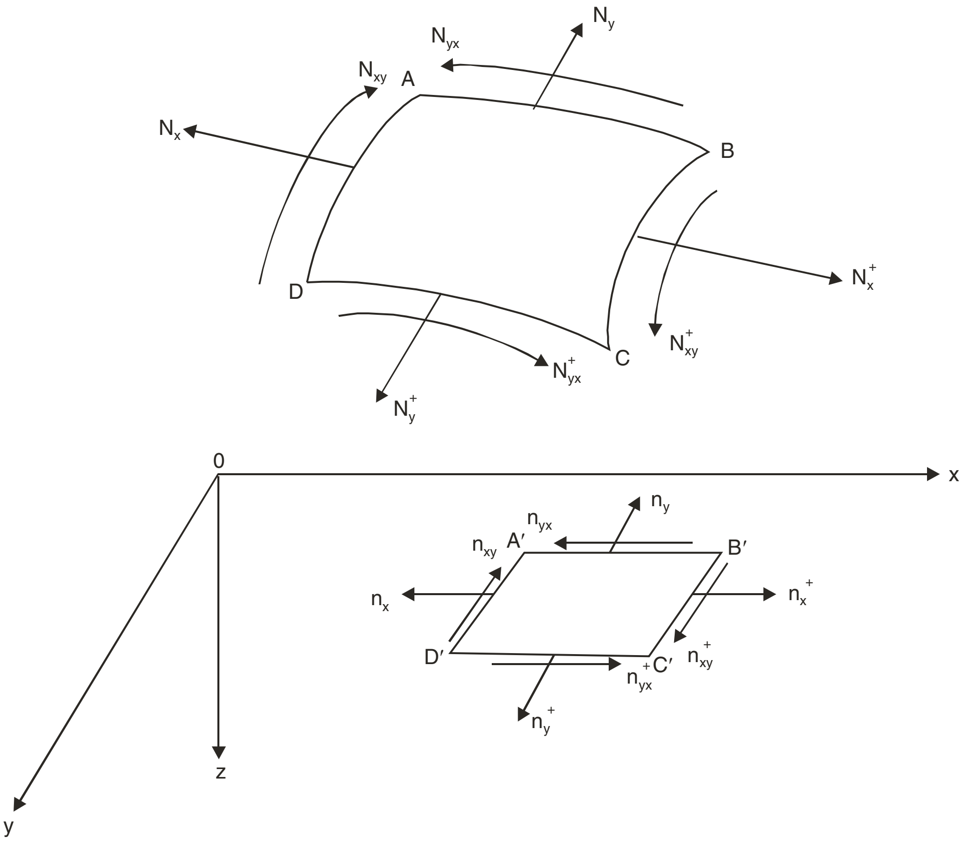


Fig. 19.3 Membrane and Pseudo forces

Relationship between Pseudo and Membrane Forces:

1. N_x and n_x forces:
   Normal force on $AD = N_x\, ds_y$
   Its component in x-direction $= N_x\, ds_y \cos\phi$
   From the definition of pseudo force,

$$n_x dy = N_x ds_y \cos\phi$$

or

$$n_x = N_x \frac{ds_y}{dy} \cos\phi$$

$$= N_x \frac{1}{\cos\psi} \cdot \cos\phi$$

$$= N_x \sqrt{1+q^2} \times \frac{1}{\sqrt{1+p^2}}$$

i.e. $$n_x = \frac{\sqrt{1+q^2}}{\sqrt{1+p^2}} N_x \qquad \text{...eqn. 19.8}$$

Similarly, $$n_y = \frac{\sqrt{1+p^2}}{\sqrt{1+q^2}} N_y \qquad \text{...eqn. 19.9}$$

2. n_{xy} and N_{xy} forces:
   Shear force on $AD = N_{xy}\, ds_y$
   Its component in y-direction $= N_{xy}\, ds_y \cos\psi$. According to the definition of the pseudo forces, this must be equal to $n_{xy} dy$.

∴ $$n_{xy}\, dy = N_{xy}\, ds_y \cos\psi$$

i.e. $$n_{xy} = N_{xy} \frac{ds_y}{dy} \cos\psi$$

$$= N_{xy} \frac{1}{\cos\psi} \cdot \cos\psi$$

i.e. $$n_{xy} = N_{xy} \qquad \text{...eqn. 19.10}$$

Similarly, $$n_{yx} = N_{yx} \qquad \text{...eqn. 19.11}$$

But $$n_{xy} = n_{yx}$$

∴ $$N_{xy} = N_{yx} \qquad \text{...eqn. 19.12}$$

Pseudo Loads:

The pseudo loads *X, Y, Z* in the directions *x, y* and *z* are so defined that

Real Load × Surface area of the element

= Pseudo load × Projected area of the element.

Hence, if F_x, F_y and F_z are intensity of load components on the element in *x, y* and *z* directions, then, according to the definition of pseudo load,

$$F_x \sqrt{1+p^2+q^2}\, dx\, dy = X\, dx\, dy$$

Thus, $$X = F_x \sqrt{1+p^2+q^2}$$

Similarly, $$Y = F_y \sqrt{1+p^2+q^2} \qquad \text{...eqn. 19.13}$$

$$Z = F_z \sqrt{1+p^2+q^2}$$

Equations of Equilibrium

Referring to the projected element $A'B'C'D'$,

Σ Forces in x-direction = 0, gives

$$\left(-n_x + n_x \frac{\partial n_x}{\partial x} dx\right) dy + \left(-n_{yx} + n_{yx} + \frac{\partial N_{yx}}{\partial y} dy\right) dx + X\, dx\, dy = 0$$

i.e.
$$\frac{\partial n_x}{\partial x} + \frac{\partial n_{xy}}{\partial y} + X = 0 \qquad \text{...eqn. 19.14}$$

Σ Forces in y-direction = 0, gives
$$\frac{\partial n_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} + y = 0. \qquad \text{...eqn. 19.15}$$

Σ Forces in z-direction = 0:

To assemble this equation, the components of various forces in z-direction are to be found.

Vertical component of N

$= N_x\, ds_y \sin\phi$, But from definition of pseudo forces $N_x\, ds_y \cos\phi = n_x\, dy$

$$= n_x \frac{dy}{\cos\phi} \sin\phi$$

$= n_x \tan\phi\, dy$

Similarly, vertical component of $N_y = n_y \tan\psi\, dx$

vertical component of $N_{xy} = n_{xy} \tan\psi\, dy$

and vertical component of $N_{yx} = n_{yx} \tan\phi\, dx$

Net vertical force of N_x - forces

$$= -n_x \tan\phi\, dy + (n_x \tan\phi)^+\, dy$$

$$= -n_x \tan\phi\, dy + n_x \tan\phi\, dy + \frac{\partial}{\partial x}(n_x \tan\phi)\, dx\, dy$$

$$= \frac{\partial}{\partial x}(n_x \tan\phi)\, dxdy$$

Similarly,

net vertical force due to N_y forces $= \dfrac{\partial}{\partial y}(n_y \tan\psi)\, dx\, dy$

net vertical force due to N_{xy} forces $= \dfrac{\partial}{\partial x}(n_{xy} \tan\psi)\, dx\, dy$

net vertical force due to N_{yx} forces $= \dfrac{\partial}{\partial y}(n_{yx} \tan\phi)\, dx\, dy$

$\therefore$ The equation of equilibrium is,

$$\frac{\partial}{\partial x}(n_x \tan\phi) + \frac{\partial}{\partial y}(n_y \tan\psi) + \frac{\partial}{\partial x}(n_{xy} \tan\psi)$$

$$+ \frac{\partial}{\partial y}(n_{yx} \tan\phi) + Z = 0$$

But
$$\tan\phi = \frac{\partial z}{\partial x} \quad \text{and} \quad \tan\psi = \frac{\partial z}{\partial y}.$$

∴ Equation of equilibrium is,

$$\frac{\partial}{\partial x}\left(n_x \frac{\partial z}{\partial x}\right)+\frac{\partial}{\partial y}\left(n_y \frac{\partial z}{\partial y}\right)+\frac{\partial}{\partial x}\left(n_{xy} \frac{\partial z}{\partial y}\right)+\frac{\partial}{\partial y}\left(n_{yx} \frac{\partial z}{\partial x}\right)+Z=0$$

i.e.

$$\frac{\partial n_x}{\partial x}\cdot\frac{\partial z}{\partial x}+n_x\frac{\partial^2 z}{\partial x^2}+\frac{\partial n_y}{\partial y}\frac{\partial z}{\partial y}+n_y\frac{\partial^2 z}{\partial y^2}+\frac{\partial n_{xy}}{\partial x}\cdot\frac{\partial z}{\partial y}$$

$$+n_{xy}\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial n_{yx}}{\partial y}\cdot\frac{\partial z}{\partial x}+n_{yx}\frac{\partial^2 z}{\partial x\partial y}+Z=0$$

Since, $n_{xy} = n_{yx}$, we get,

$$n_x\frac{\partial^2 z}{\partial x^2}+2n_{xy}\frac{\partial^2 z}{\partial x\partial y}+n_y\frac{\partial^2 z}{\partial y^2}$$

$$+\frac{\partial z}{\partial x}\left(\frac{\partial n_x}{\partial x}+\frac{\partial n_{xy}}{\partial y}\right)+\frac{\partial z}{\partial y}\left(\frac{\partial n_y}{\partial y}+\frac{\partial n_{xy}}{\partial x}\right)+Z=0$$

But from equation 1,

$$\frac{\partial n_x}{\partial x}+\frac{\partial n_{xy}}{\partial y}=-X$$

and from equation 2,

$$\frac{\partial n_y}{\partial y}+\frac{\partial n_{xy}}{\partial x}=-Y$$

∴ The equation of equilibrium is

∴ $$\mathbf{rn_x + 2sn_{xy} + tn_y = pX + qY - Z} \qquad \text{...eqn. 19.16}$$

19.6 PUCHERS STRESS FUNCTION

In 1934, Pucher introduced a stress function ϕ so as to reduce the three equations of equilibrium in three unknowns, namely n_x, n_y and n_{xy}, to one equation of equilibrium in only one unknown ϕ. The stress function ϕ is so defined that,

$$n_{xy}=-\frac{\partial^2\phi}{\partial x\partial y}$$

Then from Eqn. 19.14,

$$\frac{\partial n_x}{\partial x}=\frac{\partial^3\phi}{\partial x\partial y^2}-X$$

or

$$n_x=\frac{\partial^2\phi}{\partial y^2}-\int X\,dx$$

From equation 19.15,

$$\frac{\partial n_y}{\partial y} = \frac{\partial^3 \phi}{\partial x^2 \partial y} - Y$$

$$\therefore \qquad n_y = \frac{\partial^2 \phi}{\partial x^2} - \int Y\,dy$$

Substituting these valuation in Eqn. 19.16, we get,

$$r\left(\frac{\partial^2 \phi}{\partial y^2} - \int X\,dx\right) + 2S\left(-\frac{\partial^2 \phi}{\partial x\,\partial y}\right) + t\left(\frac{\partial^2 \phi}{\partial x^2} - \int Y dy\right)$$

$$= pX + qY - Z$$

or

$$r\frac{\partial^2 \phi}{\partial y^2} - 2S\frac{\partial^2 \phi}{\partial x \partial y} + t\frac{\partial^2 \phi}{\partial x^2} = pX + qY - Z + r\int X dx + t\int Y dy \qquad \text{...eqn. 19.17}$$

19.7 SYNCLASTIC, DEVELOPABLE AND ANTICLASTIC SHELLS

If *r, s* and *t* are curvatures, the principal curvatures $\frac{1}{R_1}$ and $\frac{1}{R_2}$ are given by

$$\frac{1}{R_1} = \frac{r+t}{2} + \sqrt{\left(\frac{r-t}{2}\right)^2 + s^2}$$

$$\frac{1}{R_2} = \frac{r+t}{2} - \sqrt{\left(\frac{r-t}{2}\right)^2 + s^2}$$

$$\therefore \qquad \frac{1}{R_1} \times \frac{1}{R_2} = \left\{\frac{r+t}{2} + \sqrt{\left(\frac{r-t}{2}\right)^2 + s^2}\right\}\left\{\frac{r+t}{2} - \sqrt{\left(\frac{r-t}{2}\right)^2 + s^2}\right\}$$

$$= \left(\frac{r+t}{2}\right)^2 - \left\{\left(\frac{r-t}{2}\right)^2 + s^2\right\}$$

$$= rt - s^2$$

A shell is synclastic, developable or anticlastic according as

$$\frac{1}{R_1 R_2} = rt - s^2 \lesseqgtr 0. \qquad \text{...eqn. 19.18}$$

19.8 MEMBRANE THEORY OF SYNCLASTIC SHELLS

Among synclastic shells, rotational paraboloid and the elliptic paraboloid are the two surfaces most frequently favoured roofs to cover very large column free rectangular or square areas.

When a convex parabola moves over another convex parabola or when a concave parabola moves over another concave parabola elliptic paraboloid surface is generated. If both parabolas are identical the surface generated is known as rotational paraboloid. The vertical sections of an elliptic paraboloid are parabolas while the horizontal sections are ellipses. Hence, the surface has the name elliptic paraboloid. The vertical sections of rotational paraboloid are parabolas while the horizontal sections are circles.

A shell roof in the form of an elliptic paraboloid or rotational paraboloid over a rectangular or square ground plan area is usually supported by shear diaphragm on all the four edges (Refer Figure 19.4). The diaphragm are assumed to be stiff enough in their own planes to receive the shell but they cannot carry any load applied normal to their plane. In other words, the diaphragm provide simple support to the shell.

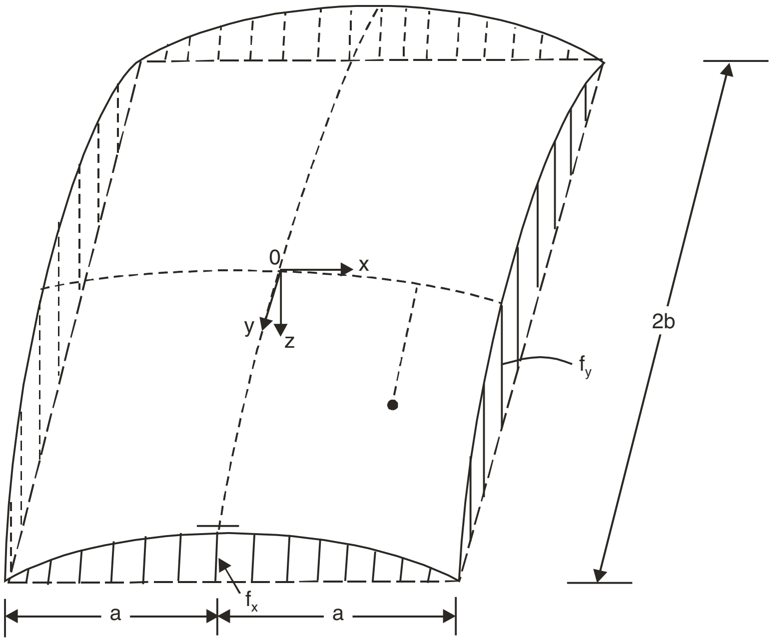


Fig. 19.4 Elliptic Paraboloid

The surface of the shell is mathematically represented by

$$Z = f_1(x) + f_2(y)$$

where $f_1(x)$ is the equation of the parabola in x-direction and $f_2(y)$ is the equation of the parabola in y-direction.

If the origin is selected at the crown of the shell (Refer Figure 19.4),

$$f_1(x) = \frac{f_x}{a^2} x^2 \text{ and } f_2(x) = \frac{f_y}{b^2} y^2.$$

∴ The equation of surface of shell is

$$z = \frac{f_x}{a^2} x^2 + \frac{f_y}{b^2} y^2 \qquad \text{...eqn. 19.19}$$

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{2f_x}{a^2}x \quad ...(a)$$

$$q = \frac{\partial z}{\partial y} = \frac{2f_y}{b^2}y \quad ...(b)$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{2f_x}{a^2} \quad ...(c)$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = 0 \quad ...(d)$$

and

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{2f_y}{b^2} \quad ...(e)$$

...eqn. 19.20

Membrane Analysis for Snow Load

Puchers equation for the shell is

$$r\frac{\partial^2 \phi}{\partial y^2} - 2s\frac{\partial^2 \phi}{\partial x \partial y} + t\frac{\partial^2 \phi}{\partial x^2} = pX + qY - Z + \int X dx + \int Y dy$$

For snow load $X = Y = 0$ and $Z = p_0$

Hence, Pucher's equation reduces to

$$\frac{2f_x}{a^2}\frac{\partial^2 \phi}{\partial y^2} + \frac{2f_y}{b^2}\frac{\partial^2 \phi}{\partial x^2} = -p_0$$

Particular integral may be assumed as,

$$\phi_1 = -\frac{a^2}{4f_x}p_0 y^2 \quad ...\text{eqn. } 19.21$$

Homogeneous solution may be written as

$$\frac{2f_x}{a^2}\frac{\partial^2 \phi}{\partial y^2} + \frac{2f_y}{b^2}\frac{\partial^2 \phi}{\partial x^2} = 0$$

Seeking the solution in the form

$\phi_2 = X\,Y$, where X is the function of x only and Y is the function of y only,

we get,

$$\frac{2f_x}{a^2}Y''X + \frac{2f_y}{b^2}YX'' = 0.$$

i.e.

$$\frac{f_y}{b^2}\frac{X''}{X} = -\frac{f_x}{a^2}\frac{Y''}{Y}$$

i.e.

$$\frac{f_y}{f_x}\frac{a^2}{b^2}\frac{X''}{X} = -\frac{Y''}{Y}$$

The above relation holds good if and only if the right hand side and left hand side expressions are equal to constant. Let the constant be λ^2. Thus,

$$\frac{f_y}{f_x}\frac{a^2}{b^2}\frac{X''}{X} = -\frac{Y''}{Y} = \lambda^2.$$

It leads to two ordinary differential equations,

$$Y'' + \lambda^2 Y = 0 \quad ...(a)$$

and

$$X'' - \frac{f_x}{f_y}\frac{b^2}{a^2}\lambda^2 = 0 \quad ...(b)$$

From equation (*a*),

$$Y = \cos \lambda y$$

and from equation (*b*),

$$X = \cosh\left(\sqrt{\frac{f_x}{f_y}}\frac{b}{a}\lambda x\right)$$

$$= \cosh \beta x$$

where

$$\beta = \sqrt{\frac{f_x}{f_y}}\cdot\frac{b}{a}\cdot\lambda$$

Keeping in mind that $N_y = \frac{\partial^2\phi}{\partial x^2}$ is to be zero at $y = \pm b$, we try a solution in the form

$$\phi_2 = -\sum_{n=1,3,...}^{\infty} A_m \cosh\beta_n x\cdot\cos\lambda_n y. \quad ...\text{eqn. } 19.22$$

where

$$\lambda_n = \frac{n\pi}{2b} \text{ and } \beta_n = \sqrt{\frac{f_x}{f_y}}\,\frac{b}{a}\cdot\frac{n\pi}{2b} = \sqrt{\frac{f_x}{f_y}}\,\frac{n\pi}{2a}.$$

∴ The total solution is

$$\phi = -\Sigma A_m \cosh\beta_n x\cdot\cos\lambda_n y - \frac{a^2}{4f_x}p_0 y^2 \quad ...\text{eqn. } 19.23$$

Hence, the expressions for the pseudo stress resultants are:

$$\left.\begin{aligned} n_x &= \frac{\partial^2\phi}{\partial y^2} = \sum_{n=1,3,...}^{\infty} A_n\lambda_n^2 \cosh\beta_n x\cos\lambda_n y - \frac{a^2}{2f_x}p_0 \\ n_y &= \frac{\partial^2\phi}{\partial x^2} = -\Sigma A_n\beta_n^2\cosh\beta_n x\cos\lambda_n y \\ n_{xy} &= -\frac{\partial^2\phi}{\partial x\,\partial y} = -\Sigma A_n\beta_n\lambda_n\sinh\beta_n x\sin\lambda_n y \end{aligned}\right\} \quad ...\text{eqn } 19.24$$

Because the traverses cannot receive any load normal to their planes, the boundary conditions to be satisfied are

$$n_x = 0 \text{ at } x = \pm a \qquad ...(1)$$

and

$$n_y = 0 \text{ at } y = \pm b. \qquad ...(2)$$

Boundary condition (2) is automatically satisfied, since, at $y = \pm b$, $\sin \lambda_n y = 0$. Boundary condition (1) helps in finding A_n. To apply this boundary condition, it is necessary to expand the uniform load p_0 in Fourier series form in the y-direction. Thus, we take,

$$p_0 = \sum_{n=1,3,\ldots}^{\infty} \frac{4p_0}{n\pi}(-1)^{\frac{n-1}{2}} \cos \lambda_n y$$

Then from boundary condition (1), we get

$$0 = \Sigma A_n \lambda_n^2 \cosh \beta_n a \cos \lambda_n y - \frac{a^2}{2f_x} \Sigma \frac{4p_0}{n\pi}(-1)^{\frac{n-1}{2}} \cos \lambda_n y$$

For all values of n,

$$A_n \lambda_n^2 \cosh \beta_n a \cos \lambda_n y = \frac{a^2}{2f_x} \frac{4p_0}{n\pi}(-1)^{\frac{n-1}{2}} \cos \lambda_n y$$

$$\therefore \qquad A_n = \frac{2p_0 a^2 (-1)^{\frac{n-1}{2}}}{\lambda_m^2 f_x n\pi \cosh \beta_n a} \qquad \text{...eqn. 19.25}$$

Hence,

$$\left.\begin{aligned} n_x &= \frac{p_0 a^2}{f_x}\left\{\frac{2}{\pi}\Sigma(-1)^{\frac{n-1}{2}} \frac{\cosh \beta_n x \cos \lambda_n y}{n \cosh \beta_n a} - \frac{1}{2}\right\} \\ n_y &= -\frac{p_0 b^2}{f_y}\left\{\frac{2}{\pi}\Sigma(-1)^{\frac{n-1}{2}} \frac{\cosh \beta_n x \cdot \cos \lambda_n y}{n \cosh \beta_n a}\right\} \\ n_{xy} &= -\frac{p_0 ab}{\sqrt{f_x f_y}}\left\{\frac{2}{\pi}\Sigma(-1)^{\frac{n-1}{2}} \frac{\sinh \beta_n x \cdot \sin \lambda_n y}{n \cosh \beta_n a}\right\} \end{aligned}\right\} \qquad \text{...eqn. 19.26}$$

The corresponding membrane stresses may be found by using the following relations:

$$\left.\begin{aligned} N_x &= x\sqrt{\frac{1+p^2}{1+q^2}}\, n_x \\ N_y &= \sqrt{\frac{1+q^2}{1+p^2}}\, n_y \\ \text{and} \qquad N_{xy} &= n_{xy}. \end{aligned}\right\} \qquad \text{...eqn. 19.27}$$

Convergence Study

Parme A.L. reported the following convergence study:

n_x and n_y converge rapidly and hence, three to four terms of the series give satisfactory accuracy. However, the expression for shear converge rather slowly at $x = \pm a$. To force n_{xy} to converge rapidly, the following recommendations has been made.

$$n_{xy}\Big|_{x=a} = -\frac{p_0 ab}{\sqrt{f_x f_y}}\left\{\frac{2}{\pi}\Sigma(-1)^{\frac{n-1}{2}}\frac{\sinh\beta_n a\sin\lambda_n y}{n\cosh\beta_n a}\right\}$$

$$= -\frac{p_a ab}{\sqrt{f_x f_y}}\left[\left\{\frac{2}{\pi}\Sigma\frac{(-1)^{\frac{n-1}{2}}}{n}\left[\frac{\sinh\beta_n a}{\cosh\beta_n a}-1\right]+1\right\}\sin\lambda_n y\right]$$

However, $$\sum_{n=1,3,\ldots}^{\infty}\frac{(-1)^{\frac{n-1}{2}}}{n}\sin\lambda_n y = \frac{1}{4}\log\left(\sec\frac{\pi y}{2b}+\tan\frac{\pi y}{2b}\right)^2$$

$\therefore$ $$n_{xy} = -p_0\frac{ab}{\sqrt{f_x f_y}}\left[\frac{1}{2\pi}\log\left(\sec\frac{\pi y}{2b}+\tan\frac{\pi y}{2b}\right) - \frac{2}{\pi}\Sigma\left(1-\tanh\beta_n a\right)\frac{(-1)^{\frac{n-1}{2}}}{n}\sin\lambda_n y\right]$$

For values $\frac{f_x}{f_y} > 1$, $\tanh\beta_n a = 1$ and hence, the second term in the expression may be ignored except perhaps for $n = 1$.

At $y = \pm b$, $\sec\frac{\pi y}{2b}$ and $\tan\frac{\pi y}{2b}$ are infinite. Hence, it indicates n_{xy} at the corner is infinite. This would be true, if the corners were completely free of normal forces and if the shell had no bending resistance. However, because of the integral action of the supporting ribs and shell, normal forces do exist at the corner. These normal forces alter the resistance to the extent that, the shear does not need to be infinite to satisfy statics. Moreover at the corner some of the load can be and is resisted by flexural resistance. From the studies made on cylindrical shells, it has been found that this flexural action is confined to a distance of approximately $0.4\sqrt{rt}$ from the rib, in which r is the radius of curvature of the shell and t is the thickness. Therefore, it is felt that the expression for shear do not apply within the distance $0.4\sqrt{rt}$ from the corner. Shear can be considered maximum at the point $y = b - 0.4\sqrt{rt}$.

19.9 MEMBRANE THEORY OF ANTICLASTIC SHELLS

Hyperbolic paraboloid and conoid are the commonly used anticlastic shells.

19.9.1 Hyperbolic Paraboloid

Geometry: If a convex parabola moves over a concave parabola or vice versa, a hyperbolic paraboloid is generated. Vertical sections of the surfaces are paraboloid and the horizontal sections are hyperbolas. Figure 19.5 shows a typical H.P. shell.

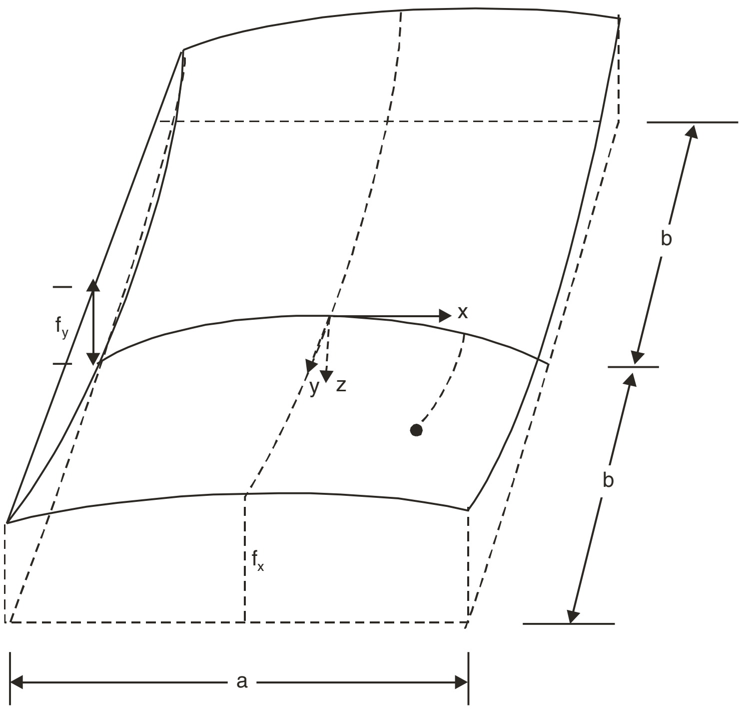


Fig. 19.5 Hyperbolic Paraboloid

Let the parabolas used be

$$z_1 = \frac{f_x}{a^2} x^2$$

$$z_2 = \frac{f_y}{b^2} y^2.$$

The geometry of the shell is defined by

$$z = -z_1 + z_2 = -\frac{f_x}{a^2} x^2 + \frac{f_y}{b^2} y^2.$$

$$\frac{\partial^2 z}{\partial x \partial y} = 0 \quad \textit{i.e. } x \text{ and } y \text{ are principal directions}$$

$$\therefore \qquad \frac{1}{R_1} = \frac{\partial^2 z}{\partial x^2} = \frac{2 f_x}{a^2}$$

$$\frac{1}{R_2} = \frac{\partial^2 z}{\partial y^2} = \frac{2 f_y}{b^2}$$

where R_1 and R_2 are the principal curvatures. Thus, in terms of principal curvatures, the surface may be defined as

$$z = -\frac{x^2}{2R_1} + \frac{y^2}{2R_2} \qquad \text{...}(a)$$

setting $z = 0$, we get

$$\left(\frac{x}{\sqrt{2R_1}} + \frac{y}{\sqrt{2R_2}}\right)\left(-\frac{x}{\sqrt{2R_1}} + \frac{y}{\sqrt{2R_2}}\right) = 0 \qquad \text{...}(b)$$

Equation (*b*) represents a pair of straight lines existing on the surface. Their inclination to *x*-axis is given by (Ref. Figure 19.6).

$$\tan\gamma = \sqrt{\frac{R_2}{R_1}}\,.$$

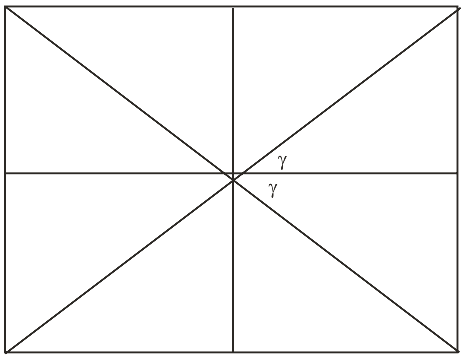


Fig. 19.6

If $R_2 = R_1$, $\tan\gamma = 1$ or $\gamma = 45°$. Hence, the angle between two pairs of straight lines is 90° *i.e.* they are orthogonal. Such surfaces are known as rectangular hyperbolic paraboloid.

If the asymptotes are chosen as coordinate axes (Ref. Figure 19.7), let x', y' be the coordinate system.

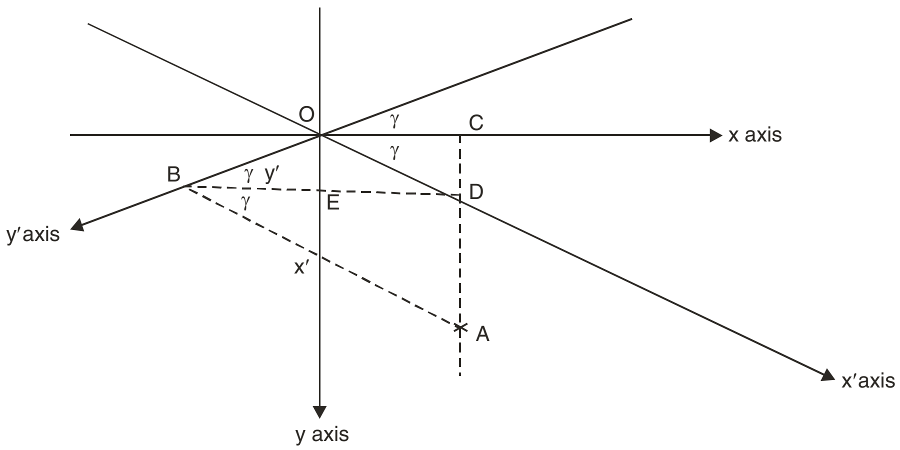


Fig. 19.7

Then looking at point A,

$$x = OC = BD - BE$$
$$= x' \cos\gamma - y' \cos\gamma$$
$$= (x' - y') \cos\gamma$$
$$y = AC = OE + AD$$
$$= y' \sin\gamma + x' \sin\gamma$$
$$= (x' + y') \sin\gamma$$

$\therefore$ Equation of hyperbolic paraboloid with respect to asymptotic axes is,

$$z = -\frac{x^2}{2R_1} + \frac{y^2}{2R_2} = -\frac{(x' - y')^2 \cos^2\gamma}{2R_1} + \frac{(x' + y')^2 \sin^2\gamma}{2R_2}$$

$$= \frac{1}{2R_2}\left[-(x' - y')^2 \cos^2\gamma \frac{R_2}{R_1} + (x' + y')\sin^2\gamma\right]$$

$$= \frac{1}{2R_2}\left[-(x' - y')^2 \cos^2\gamma \tan^2\gamma + (x' + y')^2 \sin^2\gamma\right]$$

$$= \frac{1}{2R_2}\left[-(x' - y')^2 + (x' + y')^2\right]\sin^2\gamma$$

$$= +\frac{2}{R_2} x'y' \sin^2\gamma$$

For rectangular hyperbolic paraboloid $\gamma = 45°$.

$$\therefore \qquad z = -\frac{1}{R} x'y'.$$

Looking at Fig. 19.8, it may be visualised as surface made up of straight line generaters. Then

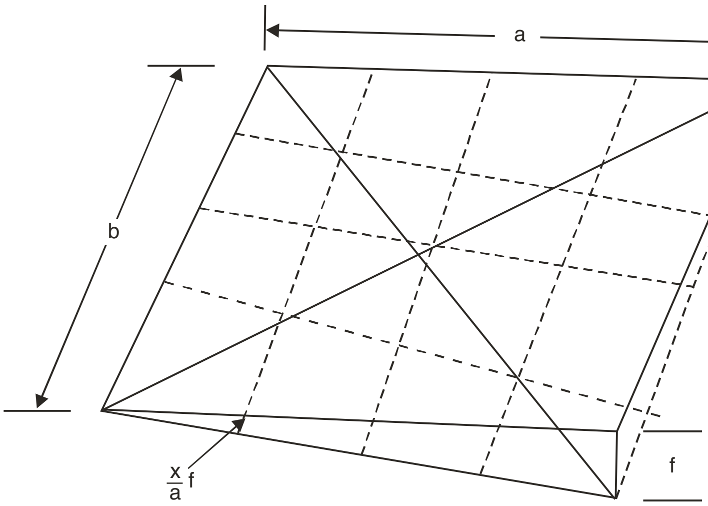


Fig. 19.8

$$Z = \frac{x}{a} f \cdot \frac{y}{b}$$

$$= xy\frac{f}{ab} = \frac{xy}{c}$$

where $c = \dfrac{ab}{f}$ is radius of curvature.

Analysis of Hyper Shells (Rectangular Hyperbolic Shells)

The shell surface is given by

$$z = \frac{xy}{c} \text{ where } c = \frac{ab}{f}.$$

$$p = \frac{\partial z}{\partial x} = \frac{y}{c},\ q = \frac{\partial z}{\partial y} = \frac{x}{c}$$

$$r = 0, \quad s = \frac{\partial^2 z}{\partial x \partial y} = \frac{1}{c} \quad t = \frac{\partial^2 z}{\partial y^2} = 0.$$

Hence, the equation of equilibrium reduces to

$$2s\ n_{xy} = -Z + pX + qY$$

For self wt:

$$X = 0, \quad Y = 0, \quad Z = g\sqrt{1 + p^2 + q^2}$$

$$\therefore \quad 2\frac{1}{c}n_{xy} = -g\sqrt{1 + \frac{x^2}{c^2} + \frac{y^2}{c^2}}$$

$$n_{xy} = -\frac{gc}{2}\sqrt{1 + \frac{x^2}{c^2} + \frac{y^2}{c^2}}$$

$$= -\frac{g}{2}\sqrt{c^2 + x^2 + y^2}$$

From equation of equilibrium 1,

$$\frac{\partial n_x}{\partial x} = -\frac{\partial n_{xy}}{\partial y} = +\frac{g}{2} \cdot 2y \times \frac{1}{2}\left(c^2 + x^2 + y^2\right)^{-1/2}$$

$$= \frac{g}{2}\frac{y}{\sqrt{c^2 + x^2 + y^2}}$$

$$\therefore \quad n_x = \frac{g}{2}\frac{\int y\,dx}{\sqrt{c^2 + x^2 + y^2}} + C.$$

$$= \frac{gy}{2}\log\left[x+\sqrt{c^2+x^2+y^2}\right]+f_1(y)$$

From equation of equilibrium 2,

$$\frac{\partial n_y}{\partial x} = -\frac{\partial n_{xy}}{\partial x} = -\frac{g}{2}\frac{(-2x)}{2\sqrt{c^2+x^2+y^2}}$$

$$= \frac{gx}{2}\log\left[y+\sqrt{c^2+x^2+y^2}\right]+f_2(x)$$

$f_1(y)$ and $f_2(x)$ are to be evaluated from the boundary conditions.

Umbrella Roof: Figure 19.9 shows a typical umbrella type H.P. Shell.

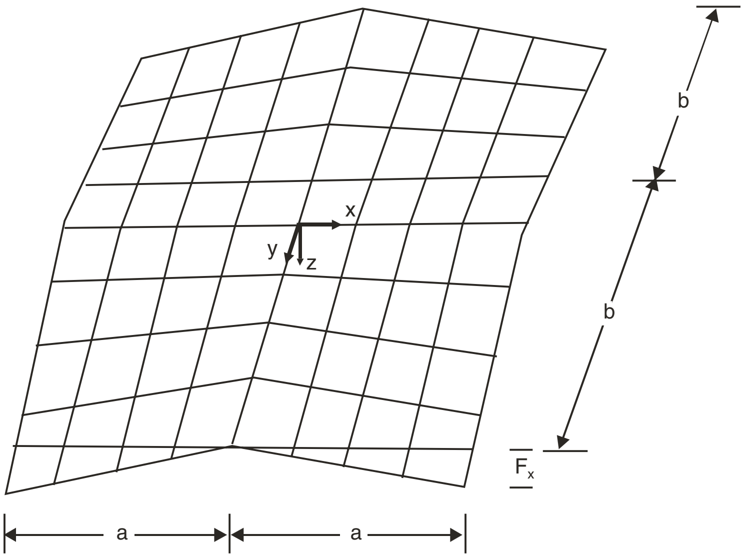


Fig. 19.9 A typical umbrella type H.P. Shell

End beams are thin and deep.

$\therefore$ $n_x = 0$ at $x = a$

and $n_y = 0$ at $y = b$.

$\therefore$
$$f_1(y) = -\frac{gy}{2}\log\left[a+\sqrt{c^2+a^2+y^2}\right]$$

$$f_2(x) = -\frac{gx}{2}\log\left[b+\sqrt{c^2+x^2+b^2}\right]$$

$\therefore$
$$n_x = \frac{gy}{2}\log\frac{x+\sqrt{c^2+x^2+y^2}}{a+\sqrt{c^2+a^2+y^2}}$$

$$n_y = \frac{gx}{2}\log\frac{y+\sqrt{c^2+x^2+y^2}}{b+\sqrt{c^2+x^2+b^2}}$$

For a shallow shell:
p^2 and q^2 are negligible compared to unity. Hence,

$$Z = g\sqrt{1+p^2+q^2} = g$$

$\therefore$ Equation of equilibrium is

$$2 \times \frac{1}{c} n_{xy} = -g$$

$$\therefore \qquad n_{xy} = -\frac{gc}{2}$$

$$n_x = 0 \text{ and } n_y = 0.$$

Thus, a shallow shell subjected to dead load only is in a state of pure shear.

Shallow and Deep shell

For a deep shell,

$$n_{xy} = -\frac{g}{2}\sqrt{c^2+x^2+y^2}$$

$$= -\frac{gc}{2}\sqrt{1+\left(\frac{x^2+y^2}{c^2}\right)}$$

$$\therefore \qquad n_{xy}\Big|_{\max} = -\frac{gc}{2}\left[1+\frac{a^2+b^2}{c^2}\right]$$

$$= -\frac{gc}{2}\left[1+\frac{1}{2}\frac{a^2+b^2}{c^2}+\ldots\right]$$

$$= -\frac{gc}{2}\left[1+\frac{1}{2}\frac{a^2+b^2}{c^2}\right]$$

Let $\quad a \le b$, so that $\quad \frac{a}{b} \le 1$

Hence, $\quad \frac{a^2+b^2}{2c^2} = \frac{a^2+b^2}{2\frac{a^2b^2}{f^2}}$

$$= \frac{1}{2}\left(1+\frac{a^2}{b^2}\right)\frac{f^2}{a^2} \le \frac{1}{2}\frac{f^2}{a^2} \text{ since } \frac{a^2}{b^2} < 1$$

Thus, if $\quad \frac{f}{a} = \frac{1}{10}, \ \frac{a^2+b^2}{2c^2} \le \frac{1}{100}$

or 1 percent of first term.

Again, if $\frac{f}{a} = \frac{1}{5}, \frac{a^2 + b^2}{2c^2} \leq \frac{1}{25}$ of first term

i.e. less than 4% of first term.

Hence, for all practical purpose, a H.P. shell may be considered shallow, if $\frac{f}{a}$ is less than or equal to $\frac{1}{5}$.

Example. Design a 10 m × 12 m invested umbrella type H.P. shell.

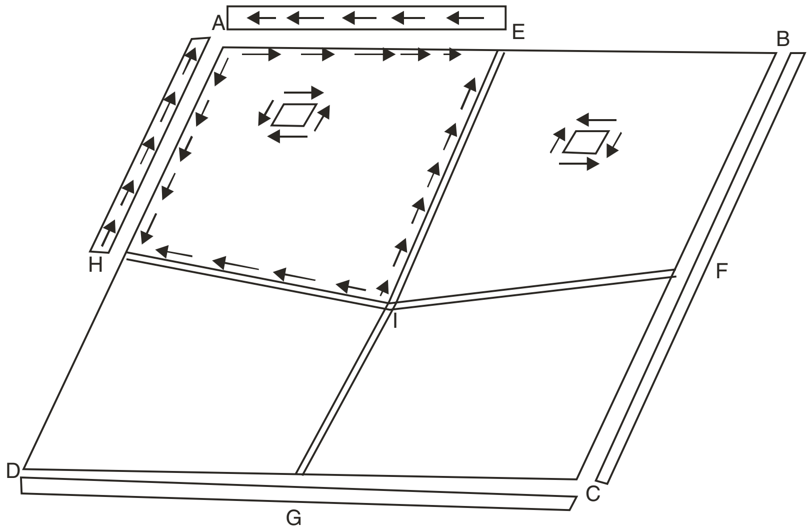


Fig. 19.10

Figure 19.10 shows a typical invested umbrella type of H.P. Shell. In this shell,

$$a = 5 \text{ m and } b = 6 \text{ m.}$$

To make it shallow amount of pulling down of a corner is kept $\frac{1}{5}$th to $\frac{1}{7}$th of least dimension *i.e.* 'a'. Hence, in this case,

$$f = 1\text{m to } \frac{5}{7} \text{ m.}$$

Let it be 1m.

$\therefore$ Radius of curvature $c = \frac{ab}{f} = \frac{5 \times 6}{1} = 30$ m.

Thickness of the roof is kept between $\frac{1}{400}$ to $\frac{1}{500}$th of radius of curvature. Thus,

$$t = \frac{1}{400} \times 30 \times 1000 \text{ to } \frac{1}{500} \times 30 \times 1000 = 75 \text{ mm to } 60 \text{ mm}.$$

Let $t = 60$ mm.

Loads:

Dead load = 0.060 × 1 × 1 × 25 = 1.5 kN/m^2

Live load: In this case, slope of line joining springing point and the crown is $\alpha = \tan^{-1} \frac{1}{\sqrt{5^2 + 6^2}} < 10°$.

∴ L.L = 0.75 kN/m^2

Let total load on shell (including finishing load) be 2.4 kN/m^2.

Since, the shell is shallow, it is subjected to pure shear,

$$n_{xy} = -\frac{gc}{2} = -\frac{2.4 \times 30}{2} = 36 \text{ kN/m}.$$

Pure shear produces tensile/compressive stresses in diagonal directions, the magnitude being the same (36 kN/m).

$$\therefore \quad A_{st} = \frac{36 \times 1000}{150} = 240 \text{ mm}^2,$$

Using 8 mm diameter bars,

$$\text{spacing} = \frac{\frac{\pi}{4} \times 8^2}{240} \times 1000 = 209 \text{ mm}.$$

Provide 8 mm bars at 200 mm c/c. Actually these bars are required in the direction of tensile stresses (45° to x axis) and distribution bars are required at 90° to that direction. In these directions, it is difficult to bend the bars. Hence, usually reinforcement is provided in x and y directions in which bars are to be straight. Provide 8 mm bars at 200 mm c/c in x and y directions so that the component of steel at 45° is sufficient to take diagonal tension.

Design of Peripheral Edge Beam

Along the edges of the shell, thin but sufficiently deep edge beams are to be provided to take up shear from the shell and transfer the load to columns. Figure 19.11(*a*) shows peripheral edge beams with the load transferred by shell and Figure 19.11(*b*) shows the transfer of load by beams connecting peripheral edge beam and columns. It may be noted that peripheral edge beams are subjected to tensile forces and the beams connected to column are subjected to compressive stresses, the magnitude being same as shear in the shell (36 kN/m).

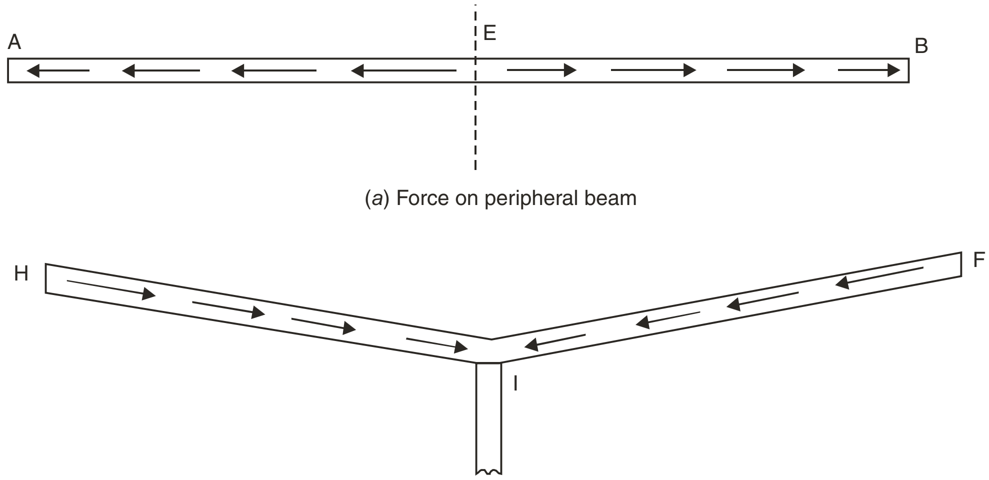
Maximum tension in peripheral beam is at section B in 6 m beam.

$$T = 36 \times 6 = 216 \text{ kN}.$$

$$A_{st} = \frac{216 \times 1000}{150} = 1440 \text{ mm}^2.$$

Provide 8 bars of 16 mm diameter.

Provide nominal shear reinforcement. Use same section for 5 m beams also.



(*a*) Force on peripheral beam

(*b*) Force on beam connecting peripheral beam and column

Fig. 19.11

Edge Beam Dimension

To control crack width, maximum permissible direct tension in M: 25 concrete = 1.3 N/mm^2.

for M: 25 concrete, $m = \dfrac{280}{3 \times 8.5} \approx 11.$

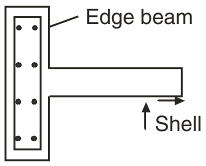
Hence, area of concrete required is

$$1.3 = \frac{216 \times 1000}{A + (11-1) \times 1440}$$

$$A = 151753 \text{ mm}^2.$$

Provide, 225 × 675 mm beams.

These edge beams should be connected to shell symmetrically so as to avoid bending.



Design of Beams Connecting Edge Beams and Column

Length of beam $= \sqrt{6^2 + 1^2} = 6.00276$ m.

∴ Maximum compressive force = 36 × 6.00276

= 218.98 kN.

From architectural point of view, the size of beam is kept same as peripheral beam.

= 225 × 675 mm.

Compressive stress in concrete

$$= \frac{218.98 \times 1000}{150 \times 675} = 2.16\,\text{N/mm}^2.$$

Hence, concrete alone can resist it. Provide minimum reinforcement.

$$A_{sc} = \frac{0.8}{100} \times 225 \times 675 = 1215 \text{ mm}^2.$$

Provide 6 bars of 16 mm diameter. Use same section for 5 m beams also.

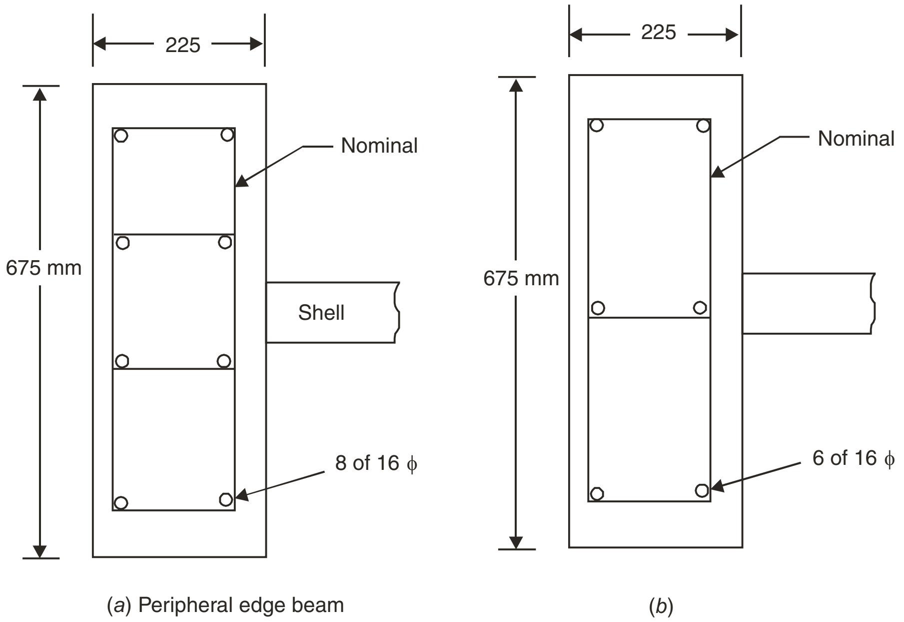


Fig. 19.12

The sections are shown in Figure 19.12.

19.9.2 Membrane Analysis of Conoid

Figure 19.13 shows a typical conoid. It may be generated by moving a straight line with one of its end on a straight line and the other end on a plane curve, keeping the line parallel to itself on a projected plane. The projected plane is known as director plane. The plane curve and the straight line are known as directrices. The conoid is inefficient in transferring the load by membrane action near straight line directrix. Hence, many times truncated conoids (Fig. 19.14) are used.

Geometry of Conoid: The plane curve used as a directrix may be circular, parabolic or catenary. Accordingly the conoids are known as circular, parabolic and catenary conoids. Of these the commonly used conoid is the parabolic.

The glazing provided at plane curve end may be vertical or inclined as shown in Fig. 19.15. In this article, analysis is carried out for type–I parabolic conoid.

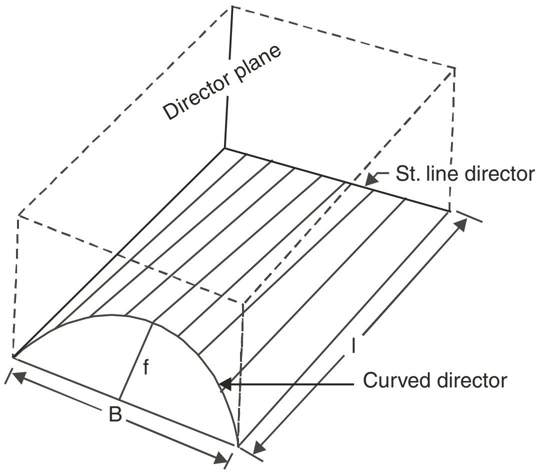


Fig. 19.13 A conoid

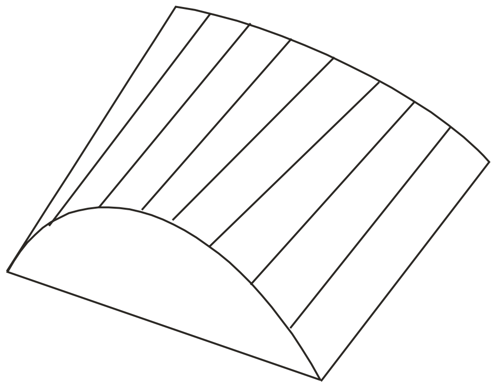


Fig. 19.14 Truncated conoid

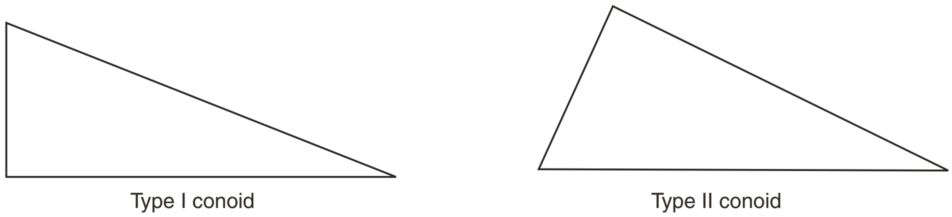


Fig. 19.15 Types of Conoids

The equation of type–I parabolic conoid, with *x, y, z,* axes selected as shown in Figure 19.16.

$$z = -\frac{4f}{B^2}\left(\frac{B^2}{4} - y^2\right)\frac{x}{l}$$

$$\therefore \qquad p = \frac{\partial z}{\partial x} = -\frac{4f}{lB^2}\left(\frac{B^2}{4} - y^2\right) = -\frac{a}{2}\left(\frac{B^2}{4} - y^2\right)$$

$$q = \frac{\partial z}{\partial y} = \frac{8f\,xy}{lB^2} = axy$$

$$r = \frac{\partial^2 z}{\partial x^2} = 0$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{8fy}{lB^2} = ay$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{8f}{lB^2}x = ax \text{ where } a = \frac{8f}{lB^2}$$

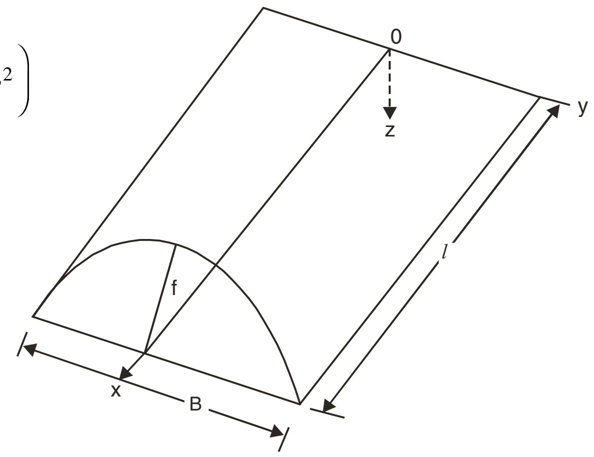


Fig. 19.16 Geometry of Parabolic Conoid of type–I

Analysis for Snow Load

All cross sections of shell are parabolas. For snow load parabola is a funicular shape. Hence, for snow load the shell degenerates into a series of independent arches. In other words, $n_x = n_{xy} = 0$. Hence, the equilibrium equation assumes the form

$$tn_y = -p_0$$

or

$$n_y = -\frac{p_0}{t}$$

i.e.

$$n_y = -p_0 \frac{lB^2}{8f_x} = -\frac{p_0}{ax}$$

∴ An appropriate stress function is

$$\phi_0 = -\frac{p_0}{a} x(\log x - 1)$$

Analysis for Dead Load

If dead load is g/unit surface area,

$$X = 0,\ Y = 0,\ Z = g\sqrt{1 + p^2 + q^2}$$

$$\approx g\left[1 + \frac{1}{2}\left(p^2 + q^2\right)\right]$$

i.e.

$$Z = g\left[1 + \frac{1}{2} \times \frac{a^2}{4}\left(\frac{B^2}{4} - y^2\right)^2 + \frac{1}{2}a^2x^2y^2\right]$$

$$= g\left[1 + a^2\left\{\frac{B^4}{8\times16} - \frac{2B^2y^2}{8\times4} + \frac{y^4}{8} + \frac{1}{2}a^2x^2y^2\right\}\right]$$

$$= g\left[1 + \frac{a^2B^4}{8\times16}\right] + ga^2\left[\frac{y^4}{8} - \frac{1}{16}B^2y^2 + \frac{x^2y^2}{2}\right]$$

$$= g\left[1 + \frac{64f^2}{l^2B^4}\frac{B^4}{8\times16}\right] + ga^2\left[\frac{y^4}{8} - \frac{1}{16}B^2y^2 + \frac{x^2y^2}{2}\right]$$

$$= g\left[1 + \frac{1}{2}\frac{f^2}{l^2}\right] + ga^2\left[\frac{y^4}{8} - \frac{1}{16}B^2y^2 + \frac{x^2y^2}{2}\right]$$

Hence, from equation of equilibrium, we get

$$2an_{xy} + axn_y = -g\left[1 + \frac{1}{2}\frac{f^2}{l^2}\right] - ga^2\left[\frac{y^4}{8} - \frac{1}{16}B^2y^2 + \frac{x^2y^2}{2}\right]$$

The above equation in terms of the stress function ϕ is,

$$-2ay\frac{\partial^2\phi}{\partial x\partial y}+ax\frac{\partial^2\phi}{\partial x^2}=-g\left[1+\frac{1}{2}\frac{f^2}{l^2}\right]-ga^2\left[\frac{y^4}{8}-\frac{B^2y^2}{16}+\frac{x^2y^2}{2}\right]$$

First part of the load term is constant and may be regarded similar to snow load. Hence, the solution for the first part of load is

$$\phi_1=-\frac{g}{a}\left[1+\frac{f^2}{2l^2}\right]x(\log x-1)$$

For the second part of the load let the stress function be ϕ_2.

Hence, $$-2ay\frac{\partial^2\phi_2}{\partial x\partial y}+ax\frac{\partial^2\phi_2}{\partial x^2}=-ga^2\left[\frac{y^4}{8}-\frac{B^2y^2}{16}+\frac{x^2y^2}{2}\right]$$

i.e. $$-2y\frac{\partial^2\phi_2}{\partial x\partial y}+x\frac{\partial^2\phi_2}{\partial x^2}=-ga\left[\frac{y^4}{8}-\frac{B^2y^2}{16}+\frac{x^2y^2}{2}\right]$$

The above equation is satisfied by a stress function of the form

$$\phi_2=\sum A_{mn}x^m y^n$$

or $$\phi_2=\sum A_{mn}\left(x^m-l^m\right)y^n$$

The latter form is preferred as it can be made to satisfy the boundary condition at $x = l$ easily. Using chosen stress function, we get,

$$\sum -2y\,A_{mn}mn\,x^{m-1}\,y^{n-1}+x\,A_{mn}m(m-1)\,x^{m-2}\,y^n=-ga\left[\frac{y^4}{8}-\frac{1}{16}B^2y^2+\frac{x^2y^2}{2}\right]$$

i.e. $$A_{mn}x^{m-1}y^n m(-2n+m-1)=-ga\left[\frac{y^4}{8}-\frac{B^2y^2}{16}+\frac{x^2y^2}{2}\right]$$

The values of A_{mn} for each of the load term may be found seperately and total solution obtained. For first term:

$$A_{mn}x^{m-1}y^n\,m(-2n+m-1)=-ga\frac{y^4}{8}$$

Comparing the powers of x and y, we get

$$m-1=0 \quad i.e. \quad m=1$$

and $$n=4.$$

$\therefore$ $$A_{mn}\times 1(-2\times 4+1-1)=-\frac{ga}{8}$$

or $$A_{mn}=\frac{ga}{64}$$

$\therefore$ Solution for first term of load is

$$\phi_2=\frac{ga}{64}(x-l)\,y^4=-\frac{ga}{64}(l-x)\,y^4$$

Similarly for second term of load

$$m = 1 \text{ and } n = 2$$

$$\therefore \qquad A_{mn} \times 1(-2 \times 2 + 1 - 1) = \frac{ga}{16} B^2$$

$$\text{or} \qquad A_{mn} = -\frac{ga}{64}(x-l) B^2$$

$$\text{Hence,} \qquad \phi_2 = -\frac{ga}{64}(x-l) B^2 x y^2 = \frac{ga}{64}(l-x) B^2 xy^2$$

For third term of load,

$$m = 3 \text{ and } n = 2.$$

$$\therefore \qquad A_{mn} \times 3(-2 \times 2 + 3 - 1) = -\frac{1}{2} ga$$

$$\text{or} \qquad A_{mn} = \frac{1}{12} \times ga$$

$$\phi_2 = \frac{ga}{12}\left(x^3 - l^3\right) y^2$$

$$= -\frac{ga}{12}\left(l^3 - x^3\right) y^2.$$

$\therefore$ The solution is,

$$\phi_2 = -\frac{ga}{64}(l-x) y^4 + \frac{gaB^2}{64}(l-x) y^2 - \frac{ga}{12}\left(l^3 - x^3\right) y^2$$

The total solution for self weight is

$$\phi = \phi_1 + \phi_2$$

$$= -\frac{g}{a}\left[1 + \frac{f^2}{2l^2}\right] x(\log x - 1) - \frac{ga}{64}(l-x) y^4 + \frac{gaB^2}{64}(l-x) y^2 - \frac{ga}{12}\left(l^3 - x^3\right) y^2$$

$$n_x = \frac{\partial^2 \phi}{\partial y^2} = -ga\left[\frac{3}{16}(l-x) y^2 - \frac{B^2}{32}(l-x) + \frac{1}{6}\left(l^3 - x^3\right)\right]$$

$$n_y = \frac{\partial^2 \phi}{\partial x^2} = \frac{ga}{2} xy^2 - \frac{g}{ax}\left[1 + \frac{f^2}{2l^2}\right]$$

$$n_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = ga\left[\frac{y^3}{16} - \frac{yB^2}{32} + \frac{x^2 y}{2}\right]$$

It may be easily verified that the boundary conditions $n_x\big|_{x=l} = 0$ is satisfied. The boundary condition $n_{xy} = 0$ along $y = 0$ (due to symmetry) is also satisfied. Since, conoid is of hyperbolic type, no boundary conditions may be prescribed on the edges $x = \pm B/2$ which are (open boundaries).

QUESTIONS

1. Explain the term 'Pseudo forces' and derive the relationship between pseudo forces and membrane forces. Use Monge's notations.
2. Derive the equations of equilibrium for the analysis of doubly curved shells. Use Pseudo forces and Puchers stress function.
3. State the Pucher's equation of equilibrium for the analysis of doubly curved shell. Derive membrane solution for a shell subject to snow load only.
4. Find the membrane solution for a rectangular hyperbolic parabola subject to self weight only. Show that if the ratio of the amount of pulling down of a corner to shorter side is less than or equal to $\frac{1}{5}$, it may be treated as shallow shell.
5. Design an inverted umbrella type H.P. shell to cover an area of 12 m × 15 m.
6. Differentiate between full conoid and truncated conoid and explain the relative merits and demerits. Discuss the need for a bending theory of conoids.

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